Projet de Recherche – The Motion of Point Vortices

Overview of the model and its applications on domains

Université Libre de Bruxelles

Lorenzo Carletti

Sous la supervision de
M. Denis Bonheure

Décembre 2020

Preface

The interest held by the analysis of an incompressible fluid's motion such as water has led it to be studied for many years. The equations modeling the evolution of such a system pose both mathematical and physical problems, and they have been approached from different angles.

The following work will focus on a particular vision of those equations, that is we study the evolution of point vortices on the surface of a non-viscous fluid. We will set ourselves in this restricted framework of 2D trajectories, starting by deriving the motion equations of those vortices. First, we will have a more intuitive but naive approach, and then we will spend some time showing rigorously that these equations do fit in the mathematical formalism of the model. In the next chapter we will look at the equations in more detail, through several examples in the plane, as well as with solid boundaries or on more general curved surfaces.

The goal has been throughout to maintain a certain mathematical rigour and to present valid reasoning, but the emphasis has also been placed on the intuition behind the different results. All the examples whether support arguments or are used to present a new phenomenon, and would all deserve a deeper investigation in a consecrated work. All figures shown here are of personal production, using numerical methods on the equations presented.

Previous knowledge of the subject is not required to follow the demonstrations, only the understanding of basic arguments in partial differential equations and more broadly Calculus. Having had an introduction to classical analytical mechanics methods may also help to understand some explanations.

This submission was written as part of the course MathF-430 of the Master in Mathematics at the Université Libre de Bruxelles. It does not claim to be exhaustive on the topics covered, and the reader is naturally invited to refer to the sources. In particular, *Mathematical theory of incompressible nonviscous fluids* (Marchioro & Pulvirenti, 1994) has a deeper mathematical analysis of the subject, it has been the main source of this work. I will also refer to *Vorticity and incompressible flow* (Majda & Bertozzi, 2002) for a greater introduction to the use of vorticity in the context of fluid dynamics, and The *N-vortex problem* (Newton, 2001) for a study of the chaos emerging from the equations, that is not discussed here.

Finally I would like to thank my supervisor for this work, M. Denis Bonheure.

Lorenzo Carletti

Contents

1 Derivation of the motion equation **1**
1.1 First Approach . . . 1
1.1.1 The equation . . . 1
1.1.2 Some existence theorems . . . 5
1.2 The Formal Derivation . . . 10
1.2.1 Motivation . . . 10
1.2.2 Theorems . . . 11

2 Applications on domains **21**
2.1 Exact Solutions . . . 21
2.1.1 Two-bodies motion . . . 21
2.1.2 Three-bodies motion . . . 22
2.2 Domains With Boundaries . . . 24
2.2.1 General theory . . . 24
2.2.2 Method of images . . . 27
2.3 Curved Surfaces . . . 39
2.3.1 Deriving the equations . . . 39
2.3.2 Some solutions . . . 43

References **45**

Chapter 1

Derivation of the motion equation

1.1 First Approach

This section is somewhat of an introduction to the model we will use throughout the project. We will start by reminding some key concepts for the study of vortex motion in a non-viscous fluid, and we will try to naively derive the main equation. We will then introduce some of the formulas needed for more rigorous proof. This section is based on Miot (2015); Marchioro and Pulvirenti (1994).

1.1.1 The equation

Vorticity-Stream Formulation : The evolution of an incompressible fluid in $\mathbb{R}^n$ is driven by the Navier-Stokes equations

$$\begin{aligned} &\frac{D}{Dt}\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} \qquad (x,t) \in \mathbb{R}^n \times [0,\infty), \\ &\operatorname{div}\mathbf{u} = 0, \end{aligned} \tag{N-S}$$

where $\mathbf{u}(x,t) = \left(u^1, u^2, \dots u^n\right)$ is the speed of the fluid, $p(x,t)$ is the scalar pressure function and $\nu \geq 0$ is the viscosity. We also define

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} u^i \frac{\partial}{\partial x^i} \tag{1.1}$$

the total derivative along a particle path.

Thereafter we will consider the fluid to be non-viscous, the vector velocity function evolves according to Navier-Stokes with $\nu = 0$, called Euler Equation. We will only be interested in the 2D equation.

Definition 1.1. The *vorticity* function of the fluid is defined as

$$\omega(x,t) = \frac{\partial u^2}{\partial x^1} - \frac{\partial u^1}{\partial x^2}.$$

This is a scalar field, and we can show (see for example Majda & Bertozzi, 2002) that its evolution is given by the Euler Equation for vorticity, and equivalent to solving Equation (N-S).

For a profile given at $t = t_0$, we have to solve the initial value problem for the Euler Equation

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0, \\ \operatorname{div} \mathbf{u} = 0, \\ \omega(t_0, x) = \omega_0. \end{cases} \tag{1.2}$$

The condition $\operatorname{div} \mathbf{u} = 0$ translates the incompressibility of the fluid. This yields the definition of a new function:

Definition 1.2. The scalar *stream function* $\psi(x,t)$ is such that

$$\mathbf{u} = \left(-\frac{\partial \psi}{\partial x^2}, \frac{\partial \psi}{\partial x^1} \right) =: \nabla^\perp \psi = (\nabla \psi)^\perp,$$

where we write for any vector $\mathbf{u} = (u^1, u^2)$, $\mathbf{u}^\perp = (-u^2, u^1)$.

Then for a given vorticity profile $\omega(x,t)$ that solves the Equation (1.2), the stream function must obey the Poisson equation

$$\omega = \frac{\partial^2 \psi}{\partial x^1 \partial x^1} + \frac{\partial^2 \psi}{\partial x^2 \partial x^2} = \Delta \psi \tag{1.3}$$

Generalisation to distributions : It will be useful to write also a weak form of the preceding equations. We would like to allow the vorticity profile at each instant to be given by a measure

$$\omega_t = \sum_{i=1}^{N} \gamma_i \delta_{z_i(t)}, \tag{1.4}$$

with δ_x the Dirac measure concentrated at x. This would be the vorticity given by a system of N localised points where all the vorticity concentrates, called *point vortices* each of intensity γ_i. The whole vorticity function will then be determined by the evolution of those points, following trajectories $z_i(t)$ in the plane that obey some other equation – that we are trying to find.
This form of ω is a distribution, and all the partial space derivatives in the Equation (1.2) and Equation (1.3) are to be taken in the weak sense and tested against some smooth compactly supported function $f : \mathbb{R}^2 \to \mathbb{R}$. As usual, we identify a function with the distribution it yields, so we will write $\omega(x,t)$ knowing that it is an abuse of notation defined against a test function as

$$\omega_t(f) = \int_{\mathbb{R}^2} f(x) \omega(x,t) dx \tag{1.5}$$

when this integral makes sense.

Suppose we have two related regular solutions, ω smooth and with compact support and $\mathbf{u}$ bounded onto $\mathbb{R}^2 \times \mathbb{R}_+$ with $\lim_{|x| \to \infty} \mathbf{u}(x,t) = 0$. The Poisson Equation for ψ leads us to introduce the fundamental solution of the Laplacian in the plane $\mathbb{R}^2$,

$$G(x) = \frac{1}{2\pi} \log |x|, \quad \forall\, x \in \mathbb{R}^2 \setminus \{0\}. \tag{1.6}$$

The solution $\psi(x,t)$ will be given by

$$\psi(x,t) = G \star \omega_t = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x-y| \, \omega(y,t) dy \tag{1.7}$$

with $\star$ the convolution product between distributions. Therefore ψ has still to be considered as a distribution, tested against $f \in C_0^\infty(\mathbb{R}^2)$.

Definition 1.3. In a general domain $D \subset \mathbb{R}^2$, we call $G_D(x;y)$ the *Green function* on D such that the solution $u(x)$ of the Poisson equation

$$\begin{cases} \Delta u = f & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

is given by

$$u(x) = \int_D G_D(x;y) f(y) dy.$$

The application of the kernel on f.

Here on the whole plane we have $G_{\mathbb{R}^2}(x;y) = \frac{1}{2\pi} \log|x-y|$. By the definition of $\mathbf{u} = \nabla^\perp \psi$ ($\nabla^\perp := \left(-\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^1}\right)$), the fact that for distributions

$$\frac{\partial}{\partial x^i}(G \star \omega_t) = \left(\frac{\partial}{\partial x^i} G\right) \star \omega_t \tag{1.8}$$

and finally that

$$\nabla^\perp G(x) = (\nabla G)^\perp(x) = \frac{x^\perp}{2\pi |x|^2} := \mathbf{K}(x) \tag{1.9}$$

we get the well known Biot-Savart Law

$$\mathbf{u}(x,t) = \left((\nabla G)^\perp \star \omega_t\right)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y,t) dy. \tag{1.10}$$

The weak version of the Euler equation, i.e. applied on a test function f, can then be written (remembering that by definition $\frac{\partial}{\partial x}\omega_t(f) = -\omega_t(\frac{\partial f}{\partial x})$):

$$\frac{d\omega_t}{dt}(f) = \omega_t(\mathbf{u}(\cdot,t) \cdot \nabla f) \qquad \mathbf{u}(\cdot,t) = \mathbf{K} \star \omega_t \tag{1.11}$$

The Motion Equation : We realise that in dimension 2 the Equation (1.2) (Euler vorticity equation) is a transport equation.

Definition 1.4 (Flow of u). The flow of the vector field $\mathbf{u}$ is defined as

$$X : \begin{array}{ccccccc} \mathbb{R}_+ \times & \mathbb{R}_+ \times & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (t, & s, & \alpha) & \mapsto & X(t,s,\alpha) \end{array} \quad \text{with}$$

$$\begin{cases} X(s,s,\alpha) = \alpha & \forall\ \alpha \in \mathbb{R}^2 \\ \frac{dX}{dt}(t,s,\alpha) = \mathbf{u}(X(t,s,\alpha),t) & \forall\ t,s \in \mathbb{R}_+. \end{cases}$$

If we follow the trajectory of a single particle in the velocity field starting in $t=0$ at $x=\alpha$, it will be described $\forall t$ as $X(t,0,\alpha)$, called integral curve of the vector field $\mathbf{u}$. This is referred to as the Lagrangian formulation of the equations.

The flow function has the properties that $\forall\ s \in \mathbb{R}_+$, $X(s,s,\alpha) = \alpha$ and that $X(t,s,\alpha) = x \Leftrightarrow \alpha = X(s,t,x)$. Then we get along an integral curve that

$$\begin{aligned}\frac{d}{dt}\left(\omega(X(t,0,\alpha),t)\right) &= \frac{\partial\omega}{\partial t}\left(X(t,0,\alpha),t\right) + \sum_{i=1}^{2}\frac{dX^i}{dt}(t,0,\alpha).\frac{\partial\omega}{\partial x^i}(X(t,0,\alpha),t) \\ &= \left(\frac{\partial\omega}{\partial t} + \mathbf{u}\cdot\nabla\omega\right)\left(X(t,0,\alpha),t\right) = 0.\end{aligned} \tag{1.12}$$

We say that the vorticity is transported by the integral curves, we see that it is constant along such a path.

Remark : By what is just above we would like to say that $\omega(X(t,0,\alpha),t) = \omega_0(\alpha)$ (or equivalently by the properties of the flow $\omega(x,t) = \omega_0(X(0,t,x))$) but this does not make sense, one cannot give punctual values to a distribution. However, coming back to the distributional definition of $\omega(t,x)$ we must have $\forall\ f \in C_0^\infty$

$$\int_{\mathbb{R}^2} f(x)\omega(x,t)dx = \int_{\mathbb{R}^2} f(x)\omega_0(X(0,t,x))dx \tag{1.13}$$

in the sense that

$$\omega_t(f) = \left(\omega_0(X(0,t,\cdot))\right)(f). \tag{1.14}$$

If, as we announced, we start with a vorticity $\omega_0 = \sum_{i=1}^N \gamma_i\delta_{z_{i,0}}$ we have at $t = 0$

$$\begin{aligned}\mathbf{u}(x,0) = \mathbf{K}\star\omega_0(x) &= \frac{1}{2\pi}\sum_i \gamma_i\mathbf{K}(x - z_{i,0}) \\ &= \frac{1}{2\pi}\sum_i \frac{(x-z_{i,0})^\perp}{|x-z_{i,0}|^2}.\end{aligned} \tag{1.15}$$

$\mathbf{u}$ becomes singular whenever $x \to z_{i,0}$ for some i, and since the vorticity starts concentrated in the points $z_{i,0}$, whose evolution depends on the velocity field at that location, the equations do not make sense even at $t = 0$. It is not clear how we can avoid this problem, but for now let us ignore it to get to the equation for the movement of the centers of the point vortices, $z_i(t)$.

By definition $z_i(t)$ follows an integral curve, $z_i(t) := X(t,0,z_{i,0})$

$$\frac{dz_i}{dt} = \frac{d}{dt}X(t,0,z_{i,0}) = \mathbf{u}\left(X(t,0,z_{i,0}),t\right) = \mathbf{K}\star\omega_t(z_i(t)) \tag{1.16}$$

Because vorticity is conserved, we know that the profile of ω_t will stay localised on the vortex points, there is no diffusion that would allow ω to spread and lose its singular form. We notice that for this form of ω_0, we have (1.13)

$$\omega_t(f) = \int f(x)\omega_0(X(0,t,x))dx = \int f(x)\sum_j \gamma_j\delta_{z_{j,0}}(X(0,t,x))dx \tag{1.17}$$

but $z_{j,0} = X(0,t,x) \Leftrightarrow x = X(t,0,z_{j,0}) = z_j(t)$

$$\Rightarrow \omega_t(f) = \sum_j \gamma_j f(X(t,0,z_{j,0})) = \sum_j \gamma_j f(z_j(t)) \quad \forall\ f \text{ test functions.} \tag{1.18}$$

We finally get, writing $\mathbf{K} \star \omega_t$ tested against some f and manipulating the variables around, that $\mathbf{u}(x,t) = \sum_j \gamma_j \mathbf{K}(x - z_j(t))$, so that the motion equation for the point vortex $z_i(t)$ is

$$\dot{z}_i(t) = \frac{1}{2\pi} \sum_{j=1, j\neq i}^{N} \gamma_j \frac{(z_i(t) - z_j(t))^\perp}{|z_i(t) - z_j(t)|^2}. \tag{MOT}$$

And the vorticity will be given $\omega(x,t) = \sum_{i=1}^N \gamma_i \delta_{z_i(t)}(x)$.

This equation and its analysis will be the main focus in the following sections, but there are some important points to raise.

Remark i: One can see that in the Equation (MOT) we have discarded the problematic contribution of $j = i$. This corresponds to claim that a vortex point does not feel its own velocity field, its movement depends only on the others'.

Remark ii: This consideration comes from the physical intuition that if the vortex point was alone, only immersed in its own field, it should stay at rest.

Remark iii: Another way to convince ourselves that this holds is to consider a sequence of smooth C_0^∞ functions $\omega_n \rightharpoonup \delta$ in the weak sense. If we assume the ω_n to be spherically symmetric, then $\mathbf{u}_n = 0$ for all n.

1.1.2 Some existence theorems

Definition 1.5. The set of configurations with collisions is defined by

$$\Theta := \left\{ (z_1, \dots z_N) \in \left(\mathbb{R}^2\right)^N \mid \exists\ i \neq j \quad \text{s.t. } z_i = z_j \right\}$$

The equation of motion is defined only outside of this set.

Since the kernel $\mathbf{K} = \frac{x^\perp}{2\pi|x|^2}$ is continuous and locally Lipschitz outside Θ, for all trajectories staying outside of Θ for $t \in \mathbb{R}_+$, we can apply the Cauchy-Lipschitz Theorem to get the first existence theorem.

Theorem 1.6 (Cauchy-Lipschitz). Consider the initial value problem

$$\begin{cases} \dot{z}_i = \dfrac{1}{2\pi} \displaystyle\sum_{j=1, j\neq i}^{N} \gamma_i \frac{(z_i - z_j)^\perp}{|z_i - z_j|^2} & i = 1, \dots N \\ z_i(0) = z_{i,0} & (z_{1,0}, \dots z_{N,0}) \in \mathbb{R}^{2N} \setminus \Theta \end{cases} \tag{1.19}$$

defined on $\mathbb{R}^{2N} \setminus \Theta$, $t \in [0, \infty)$. There exists $T > 0$ and a unique maximal solution

$$(z_1(t), \dots z_N(t)), \qquad t \in [0, T),$$

of class C^1. Finally either $T = +\infty$ or $\lim_{t\to T}(z_1, \dots z_n) \in \partial(\mathbb{R}^{2N} \setminus \Theta)$, i.e. either two particles meet in finite time and the speed $\dot{z}_i(t)$ becomes infinite or a particle goes to infinity, $z_i \to \infty$ in finite time for some i.

This tells us that choosing non-problematic initial conditions, the equation can be solved. The particles will follow some path until $t \to T$, then the evolution can no longer be described by the Equation (MOT). The problems that could arise and prevent the maximal solution to be defined for all times are when two particles meet, or if some particle reaches an infinite velocity in finite time. Global existence of trajectories is then strictly linked to the study of collisions in the system.

We notice directly that the system has a Hamiltonian structure, that means if we define

$$H : \mathbb{R}^{2N} \setminus \theta \to \mathbb{R} : (z_1, \dots z_N) \mapsto -\frac{1}{4\pi} \sum_{i,j=1; i\neq j}^{N} \gamma_i \gamma_j \log |z_i - z_j| \left(= -\frac{1}{2} \sum_{i \neq j} \gamma_i \gamma_j G_{\mathbb{R}^2}(z_i; z_j) \right) \tag{1.20}$$

called the *Hamiltonian* of the motion, then writing $z_i = (z_i^1, z_i^2)$,

$$\begin{cases} \gamma_i \dfrac{dz_i^1}{dt} = \dfrac{\partial H}{\partial z_i^2} \\ \gamma_i \dfrac{dz_i^2}{dt} = -\dfrac{\partial H}{\partial z_i^1}. \end{cases} \tag{1.21}$$

This a useful consideration as it allows us to apply several known analytical mechanics' results. First of all Liouville theorem tells us that the areas of domains are conserved when transported along by the motion. We also know that since H does not explicitly depend on t, H is a constant of motion, also called conservation law or first integral of the system. Another trivial constant is $\Gamma := \sum_i \gamma_i$. Finally, by Nöether's theorem the symmetries of the problem give rise to new conservation laws:

$$\text{Translation invariance } \rightsquigarrow \qquad M(t) := \sum_{i=1}^{N} \gamma_i z_i(t) = \Gamma m(t). \tag{1.22}$$

$$\text{if } \Gamma \neq 0 \quad m(t) := \sum_{i=1}^{N} \frac{\gamma_i}{\Gamma} z_i(t) \qquad \text{(center of vorticity)}. \tag{1.23}$$

$$\text{Rotational invariance } \rightsquigarrow \qquad I(t) := \sum_{i=1}^{N} \gamma_i |z_i(t)|^2 \qquad \text{(moment of inertia)}. \tag{1.24}$$

Proposition 1.7. Given a solution $(z_1(t), \dots z_N(t))$ of the problem (1.19), we have the following constants of motion

$$\begin{array}{ccc} \Gamma = \text{Cst}; & H(t) = H(0); & M(t) = M(0); \\ m(t) = m(0); & I(t) = I(0); & T(t) = T(0). \end{array}$$

Where $T(t) := \sum_{i,j} \gamma_i \gamma_j |z_i(t) - z_j(t)|^2$.

Proof (for T). The quantity T will be useful later, and we have

$$\begin{aligned} T(t) &= \sum_{i,j} \gamma_i \gamma_j \left|z_i(t) - z_j(t)\right|^2 \\ &= \sum_{i,j} \gamma_i \gamma_j \left(|z_i|^2 + |z_j|^2 - 2\langle z_i, z_j\rangle\right) = 2\Big[\Gamma I - \langle \sum_i \gamma_i z_i, \sum_j \gamma_j z_j\rangle\Big] \\ &= 2\left[\Gamma I - |M|^2\right], \end{aligned}$$

so that T is also constant for $t \in [0, T)$. □

These integrals alone make it possible to solve explicitly the cases $N = 2$ and $N = 3$. They also allow to derive the existence of global solutions if all the intensities have the same sign.

Theorem 1.8. Let $Z(t) = (z_1, \dots z_N)$ be the solution to the problem (1.19) with intensities all of the same sign, say $\gamma_i > 0 \ \forall\ i$. Then Z is defined on the whole $\mathbb{R}_+$, i.e. $T = +\infty$. Moreover we have a lower-bound on the closeness of two particles at all times, and an upper-bound on their location in the plane, that is $\exists\ \lambda, \Lambda > 0$ such that

$$\inf_{t\in\mathbb{R}_+} \min_{i\neq j} \left|z_i(t) - z_j(t)\right| \geq \lambda, \tag{1.25}$$

$$\sup_{t\in\mathbb{R}_+} \max_{i} \left|z_i(t)\right| \leq \Lambda. \tag{1.26}$$

Proof. Let us define

$$J(t) := T(t) - 2H(t) = \sum_{i,j} \gamma_i \gamma_j \left(\left|z_i(t) - z_j(t)\right|^2 - \log\left|z_i(t) - z_j(t)\right|\right).$$

We know that $J(t)$ is constant on $[0, T)$ since T and H are constant, and we can verify that $|x|^2 - \log|x| \geq 0$ so that

$$J(0) = J(t) \geq \gamma_i \gamma_j \left(\left|z_i - z_j\right|^2 - \log\left|z_i - z_j\right|\right) \geq -\gamma_i \gamma_j \log\left|z_i - z_j\right| \quad \forall\ i, j.$$

Reverting this equation we get

$$\left|z_i(t) - z_j(t)\right| \geq \exp\left(\frac{-J(0)}{\gamma_i \gamma_j}\right).$$

Taking the minimum of the right-hand side of the equation, i.e. since $J(0) > 0$ choosing the two smallest $\gamma_i, \gamma_j,\ i \neq j$, we found a constant $\lambda > 0$ satisfying (1.25).
On the other hand we also have

$$I(0) = I(t) \geq \gamma_i \left|z_i\right|^2,$$

which implies

$$|z_i(t)| \leq \left(\frac{I(0)}{\gamma_i}\right)^{1/2}$$

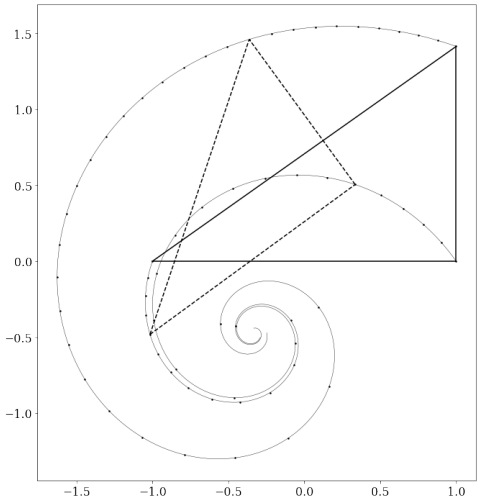


Figure 1.1: The collapse of three vortex points in a finite time.

and we get Λ satisfying (1.26).
With theses two bounds holding, we know that the velocity must remain finite at all times, and that no vortex point can tend to infinity in finite time nor tend one to another. The Cauchy-Lipschitz Theorem (1.6) then implies that $T = +\infty$ as stated. □

The situation is very different and the theorem does not hold when the vorticities have different signs, as illustrated below.

Example 1.9. In the case $N = 3$, let us observe the evolution of three point vortices with $\gamma_1 = 2, \gamma_2 = 2, \gamma_3 = -1$ starting at

$$z_{1,0} = (-1, 0), \qquad z_{2,0} = (1, 0), \qquad z_{3,0} = (1, \sqrt{2}) \tag{1.27}$$

In the case of three bodies, we can show (see Section 2.1.2, Equation (1.9)) that the evolution is described equivalently in terms of the distances between the points, if we set $l_{ij}(t) := |z_i(t) - z_j(t)|$,

$$\frac{d}{dt}(l_{ij}^2) = \frac{2}{\pi} A \gamma_k \left(\frac{1}{l_{jk}^2} - \frac{1}{l_{ki}^2} \right) \tag{1.28}$$

with i, j, k in cyclical order, $A = (z_2 - z_3) \cdot (z_3 - z_1)^\perp$ being the area of the triangle.
First of all we realise that the length ratios stay the same, for instance

$$\begin{aligned} \frac{d}{dt}\left(\frac{l_{12}^2}{l_{23}^2}\right) &= \frac{4}{\pi l_{23}^4} A \left[\gamma_3 l_{23}^2 \left(\frac{1}{l_{23}^2} - \frac{1}{l_{31}^2} \right) - \gamma_1 l_{12}^2 \left(\frac{1}{l_{31}^2} - \frac{1}{l_{12}^2} \right) \right] \\ &= \frac{4}{\pi l_{23}^4} A \Big(\gamma_1 + \gamma_3 - \frac{1}{\gamma_2 l_{31}^2} \underbrace{(\gamma_2 \gamma_3 l_{23}^2 + \gamma_1 \gamma_2 l_{12}^2)}_{= -\gamma_1 \gamma_3 l_{31}^2} \Big) \\ &= \frac{4}{\pi l_{23}^4 \gamma_2} A (\gamma_1 \gamma_2 + \gamma_2 \gamma_3 + \gamma_3 \gamma_1) = 0 \end{aligned} \tag{1.29}$$

Using hereabove that with these initial conditions $T(t) = 2.(\gamma_2 \gamma_3 l_{23}^2 + \gamma_1 \gamma_2 l_{12}^2 + \gamma_1 \gamma_3 l_{31}^2) = T(0) = 0$, $\gamma_1 \gamma_2 + \gamma_2 \gamma_3 + \gamma_3 \gamma_1 = 0$. By essentially the same calculation we get the same result for all the other ratios and this shows that the three points evolve forming self-similar triangles.

Moreover, we have

$$A = l_{23}l_{31}\sin(\alpha) = \sin(\alpha)\frac{l_{23}}{l_{12}}\frac{l_{31}}{l_{12}}l_{12}^2 = A_0 l_{12}^2 \text{ for some constant } A_0. \tag{1.30}$$

Using it in (1.28) we have

$$\frac{d}{dt}(l_{12}^2) = \frac{2}{\pi}A_0\gamma_3\left(\frac{l_{12}^2}{l_{23}^2} - \frac{l_{12}^2}{l_{31}^2}\right) = \text{Cst}, \tag{1.31}$$

which yields

$$l_{12}^2(t) = C^*t + l_{12}^2(0). \tag{1.32}$$

But here $C^* < 0$. Indeed its sign depends only on γ_3 and $\frac{l_{12}^2}{l_{23}^2} - \frac{l_{12}^2}{l_{31}^2}$ and at $t = 0$, $\frac{l_{12}^2}{l_{23}^2} - \frac{l_{12}^2}{l_{31}^2} > 0$, $\gamma_3 < 0$. This means that $l_{12}^2(t) \to 0$ in a finite time $t = \frac{-l_{12}^2(0)}{C} > 0$, the system collapses into a point (see Figure 1.1). △

Definition 1.10 (Collisions). Let $(z_1(t), \dots z_N(t))$ be a solution given by Theorem 1.6, and assume it is defined only until $T \in \mathbb{R}$, with $T < \infty$.

- With 3 bodies, if we have a system as the one above where $l_{ij} \to 0$ in finite time, and the ratios of lengths stay constant, it is said we have a *self-similar collision*;
- For N bodies, we call *partial collision* a situation where $\liminf_{t\to T} \min_{i\neq j} |z_i - z_j| = 0$;
- We call *total collision* a situation where $\liminf_{t\to T} \max_{i\neq j} |z_i - z_j| = 0$.

From now on we will say there is a collision when there is partial collision, i.e. at least two point meet in finite time.

Remark i: Regarding Example 1.9, actually there is a theorem (found in Miot, 2015, p.43) which tells us that a 3-bodies collision in finite time is necessarily self-similar.

Remark ii: Still in the Example 1.9, we observe that modifying slightly the initial conditions, setting $z_{3,0} = (1, -\sqrt{2})$ the evolution is still self-similar but $C > 0$ and the triangle grows. The solution will be global in time.

This consideration leads us to ask ourselves whether in the general case initial conditions leading to collision are frequent. To put this rigorously, let us take a measurable (for the Lebesgue measure λ on $\mathbb{R}^{2N}$) bounded subset $A \subset \mathbb{R}^{2N}$, if we define $A_{col} \subset A$ to be the set of all starting positions leading to a collision in finite time, the probability of a collapse starting inside of A is defined $\frac{\lambda(A_{col})}{\lambda(A)}$. We would like to know if this will always be negligible. For that, we introduce the concept of an ε-collapse, that will also be useful in the next section.

Definition 1.11. We define an *ε-collapse* as an event in which two point vortices get closer than ε.

We want to estimate the probability of such an event. To do that we then introduce a regularisation of the dynamic. For $0 < \varepsilon < 1$ let us define

$$\log_\varepsilon : \mathbb{R}_+ \to \mathbb{R}, \quad \in C^\infty(\mathbb{R}) \quad \text{s.t.} \quad \begin{aligned} &\log_\varepsilon |x| = \log|x| && \forall\ |x| > \varepsilon \\ &|\log_\varepsilon |x|| \le |\log|x|| && \forall\ x \in \mathbb{R} \\ &\left|\frac{d}{dx}\log_\varepsilon |x|\right| \le \frac{1}{|x|} \end{aligned} \tag{1.33}$$

and the regular Green function $G_\varepsilon(x-y) := \frac{1}{2\pi}\log_\varepsilon|x-y|$ and its associated kernel $\mathbf{K}_\varepsilon(x-y) := \nabla^\perp G_\varepsilon$.

The regularised motion equations then become

$$\begin{aligned}&\frac{d}{dt}z_i^\varepsilon(t) = \sum_{j\neq i}\gamma_j\mathbf{K}_\varepsilon(z_i^\varepsilon(t)-z_j^\varepsilon(t))\\&\text{with } z_i^\varepsilon(0) = z_{i,0}.\end{aligned} \tag{1.34}$$

This system is strictly equivalent to Equation (MOT) until the first ε-collapse, as the dynamic it describes is the same. With these tools we can prove that the probability of such a collapse is infinitesimal in ε. We write $Z_0 = (z_{1,0},\dots z_{N,0})$ the initial condition, and for $T\in\mathbb{R}$

$$d_T^\varepsilon(Z_0) := \inf_{0\leq t\leq T}\min_{i\neq j}\left|z_i^\varepsilon(t)-z_j^\varepsilon(t)\right| \tag{1.35}$$

Theorem 1.12. With the condition that $\Gamma = \sum_i\gamma_i \neq 0$, we have

$$\lim_{\varepsilon\to 0}\lambda_R\left(\{Z_0 \mid d_T^\varepsilon(Z_0) < \varepsilon\}\right) = 0,$$

where λ_R is the Lebesgue measure on the Sphere of radius R in $\mathbb{R}^{2N}$.

A proof of this can be found in Marchioro and Pulvirenti (1994, p.145). A direct implication of this theorem is thus the fact that the initial value problem (1.19) associated to the vortex motion has a global solution, for initial conditions outside a set of null Lebesgue measure. We see this taking a sequence of radii $R_n\to\infty$, for all of which we get a global flow for almost all initial conditions. The full set of all initial conditions leading to a collapse is then a countable union of sets of measure 0, and so is also negligible.

1.2 The Formal Derivation

Now that we have laid the foundations of our theory, we are interested in formally deriving the set of equations (MOT) from the Euler dynamics. In this section we will present two main theorems that will allow us to reach that goal, it is based in its totality on Marchioro and Pulvirenti (1994).

1.2.1 Motivation

Let us recall the equation describing the vortex dynamics, with initial values:

$$\begin{cases}\dot{z}_i(t) = \dfrac{1}{2\pi}\displaystyle\sum_{j=1,j\neq i}^{N}\gamma_j\frac{(z_i(t)-z_j(t))^\perp}{|z_i(t)-z_j(t)|^2}, & i=1,\dots N.\\ z_i(0) = z_{i,0},\end{cases} \tag{MOT}$$

In the last section, we found out (Theorem 1.12) that in general this initial value-problem has a global solution. The events in which the system (MOT) does not make sense are exceptional, in the sense that they only happen starting in a set of null measure. This is an important step towards proving the physical usefulness of these equations, if this system became singular for a lot of initial conditions it would be much less interesting to study its implications. As we will see in the

next chapter, these equations lead to interesting behavior of solutions (chaotic dynamic for at least $N = 4$, general stable configurations (see Miot, 2015)) and one could be satisfied with the heuristic derivation of the last section and only study it mathematically.

However, we are interested in the physical meaning of the equations and their interpretation, we would like to be sure that the vortex dynamic described by (MOT) is strictly equivalent to the dynamic governed by the Euler Equation. Indeed it is the correct physical modeling of fluid dynamics.

To be precise, if at $t = 0$,

$$\omega_0 \rightharpoonup \sum_i \gamma_i \delta_{z_{i,0}} \tag{2.1}$$

in the weak sense and we denote by ω_t the solution of the Euler Equation with initial value ω_0 and by $z_i(t)$ the solutions of the vortex equations with initial conditions $\{z_{i,0}\}$, the question is whether with (2.1) only we can infer that ω_t tends weakly to

$$\omega_t \rightharpoonup \sum_i \gamma_i \delta_{z_i(t)}, \tag{2.2}$$

or if we need new hypotheses on the physics of the system to converge to the vortex model.

The fundamental problem in the naive derivation was the removal of the self-interaction term. The justification we gave involving symmetries and physical intuition is unsatisfactory: Indeed a slight perturbation could totally destroy the symmetry, physically it is impossible to have a concentration collapsing to a point in a perfectly symmetric way. Moreover, even if we started with a symmetric initial condition, the nonlinear dynamic would break the symmetry right away.

1.2.2 Theorems

We start by considering a single vortex, immersed in a given time-dependent smooth external velocity field. This comes down to consider that the field created by the other point vortices is given. By taking ω_ε approaching a δ distribution in a particular way, we will see that in the weak sense the center of vorticity moves like a single particle in the given field without vorticity. Even though the global motion of the fluid could not converge at all due to the singularity of the kernel $\mathbf{K}$ which forces a fluid particle to rotate at great speed around the point vortex, the motion of the center is much more regular.

In the following, $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ denotes the characteristic function of any set A.

Theorem 2.1. Let us take a collection $(\Lambda_\varepsilon), \varepsilon < 1$ of open sets with

$$\lambda(\Lambda_\varepsilon) = \varepsilon^2 \qquad\qquad \Lambda_\varepsilon \subset B(x_0, \alpha\varepsilon) \quad \text{for some } \alpha > 0$$

where $B(x_0, R)$ is defined to be the disc of center x_0 and radius R. Let $\mathbf{F}$ be a divergence-free uniformly bounded time-dependent field satisfying the Lipschitz' property, with $L > 0$ such that

$$|\mathbf{F}(x,t) - \mathbf{F}(y,t)| \leq L\,|x - y|\,. \tag{2.3}$$

We then take every time an initial distribution of vorticity

$$\omega_{\varepsilon,0}(x) = \varepsilon^{-2}\chi_{\Lambda_\varepsilon}(x), \tag{2.4}$$

and write $\omega_{\varepsilon,t}$ the weak solution of Euler Equation with initial condition $\omega_{\varepsilon,0}$:

$$\frac{d}{dt}\omega_{\varepsilon,t}(f) = \omega_{\varepsilon,t}\left((\mathbf{u}_{\varepsilon,t} + \mathbf{F}) \cdot \nabla f\right) \qquad \mathbf{u}_{\varepsilon,t} = \mathbf{K} \star \omega_{\varepsilon,t} \tag{2.5}$$

with $\omega_{\varepsilon,t}$ localised in a $\Lambda_\varepsilon(t)$. Then for $T > 0$, with those assumptions,

i. $\lim_{\varepsilon\to 0} m_\varepsilon(t) = m(t)$ where $m_\varepsilon(t) = \int \omega_\varepsilon(x,t).x\ dx$
the center of vorticity of the patch at time t, and $m(t)$ is the trajectory of a particle satisfying the initial value-problem

$$\frac{dm}{dt}(t) = \mathbf{F}(m(t),t) \qquad m(0) = x_0.$$

ii. $\lim_{\varepsilon\to 0} \omega_{\varepsilon,t}(f) = f(m(t)) \qquad t \in [0,T]$

iii. $\forall\ d > 0$, $\exists\ \varepsilon_0 > 0$, depending on d, T, such that if $\varepsilon < \varepsilon_0$, $\Lambda_\varepsilon(t) \subset B(m_\varepsilon(t), d)$, i.e $\Lambda_\varepsilon(t)$ is strictly localized around $m_\varepsilon(t)$, for $t \in [0,T]$.

Remark : By the Hamiltonian nature of the Euler Equation, the energy being conserved in the absence of viscosity, $\omega_{\varepsilon,t}$ preserves the Lebesgue measure. We have $\forall\ t \geq 0\ \int \omega_\varepsilon(x,t)dx = 1$ where once again we write $\omega_\varepsilon(x,t)$ in an abuse of notation as we did earlier. We can see the distribution $\omega_{\varepsilon,t}$ as a probability density, distributed in the plane. We will then be able to apply $\omega_{\varepsilon,t}$ to any measurable function f in the sense that $\omega_\varepsilon(x,t)dx$ is a finite measure on the plane. This extends the definition of the distribution $\omega_{\varepsilon,t}$, and allows us to apply it on non-bounded functions as $x \mapsto x$, we will write thereafter

$$\omega_{\varepsilon,t}(1) = \int \omega_\varepsilon(x,t)dx = 1,$$
$$\omega_{\varepsilon,t}(X) := \int x.\omega_\varepsilon(x,t)dx = m_\varepsilon(t).$$

Proof.

Step 1. We define

$$I_\varepsilon(t) := \int \omega_\varepsilon(x,t)(x - m_\varepsilon(t))^2 dx = \omega_{\varepsilon,t}((X - m_\varepsilon(t))^2), \tag{2.6}$$

we shall start by showing that this moment of inertia vanishes for $\varepsilon \to 0$ around $m_\varepsilon(t)$. Taking the time-derivative of m_ε we get

$$\begin{aligned}\frac{d}{dt}m_\varepsilon(t) &= \frac{d}{dt}\omega_{\varepsilon,t}(X) = \omega_{\varepsilon,t}\left((\mathbf{u}_{\varepsilon,t} + \mathbf{F}) \cdot 1\right) \\ &= \omega_{\varepsilon,t}\left(\mathbf{u}_{\varepsilon,t} + \mathbf{F}\right) = \omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t}) + \omega_{\varepsilon,t}(\mathbf{F})\end{aligned}$$

since $\omega_{\varepsilon,t}$ as a measure is still linear.

$$\begin{aligned}\text{But } \omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t}) &= \int \omega_\varepsilon(x,t)\mathbf{u}_\varepsilon(x,t)dx = \int \omega_\varepsilon(x,t)(\omega_{\varepsilon,t} \star \mathbf{K})(x)dx \\ &= \int \omega_\varepsilon(x,t) \int \omega_\varepsilon(y,t)\mathbf{K}(x-y)dy\ dx.\end{aligned}$$

And since $\mathbf{K}$ is anti-symmetric, $\mathbf{K}(x-y) = -\mathbf{K}(y-x)$, we end up with

$$\omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t}) = -\iint \omega_\varepsilon(x,t)\omega_\varepsilon(y,t)\mathbf{K}(y-x)dy\ dx = -\omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t})$$

that leads to $\omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t}) = 0$. (2.7)

This means

$$\frac{d}{dt}m_\varepsilon(t) = \omega_{\varepsilon,t}(\mathbf{F}(\cdot,t)). \tag{2.8}$$

If there was no external field, both the center of vorticity and the angular momentum would be conserved as their time-derivative would be 0.

In a similar way, we take the derivative of $I_\varepsilon(t)$:

$$\begin{aligned}\frac{dI_\varepsilon}{dt}(t) &= \frac{d}{dt}\left(\omega_{\varepsilon,t}((X-m_\varepsilon(t))^2)\right) \\ &= \frac{d\omega_{\varepsilon,t}}{dt}((X-m_\varepsilon(t))^2) - 2\omega_{\varepsilon,t}\left(\frac{dm_\varepsilon}{dt}\cdot(X-m_\varepsilon(t))\right),\end{aligned}$$

$$\begin{aligned}\text{but } \frac{d\omega_{\varepsilon,t}}{dt}((X-m_\varepsilon(t))^2) &= \omega_{\varepsilon,t}\left((\mathbf{u}_{\varepsilon,t}+\mathbf{F})\cdot\nabla(X-m_\varepsilon(t))^2\right) \\ &= 2\ \omega_{\varepsilon,t}\left((\mathbf{u}_{\varepsilon,t}+\mathbf{F})\cdot(X-m_\varepsilon(t))\right) \\ &= 2\ \omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t}\cdot(X-m_\varepsilon(t))) + 2\ \omega_{\varepsilon,t}(\mathbf{F}\cdot(X-m_\varepsilon(t)))\end{aligned}$$

$$\text{with } \omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t}\cdot(X-m_\varepsilon(t))) = \omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t}\cdot X) - \omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t}\cdot m_\varepsilon(t)).$$

But by linearity and (2.7),

$$\begin{aligned}\omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t}\cdot m_\varepsilon(t)) &= m_\varepsilon(t)\cdot\omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t}) = 0, \\ \text{and } \omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t}\cdot X) &= \int \omega_\varepsilon(x,t)\mathbf{u}_\varepsilon(x,t)\cdot x\ dx \\ &= \int \omega_\varepsilon(x,t)\int \mathbf{K}(x-y)\omega_\varepsilon(y,t)\cdot x\ dx\ dy \\ &= \iint \omega_\varepsilon(x,t)\omega_\varepsilon(y,t)\mathbf{K}(x-y)\cdot x\ dx\ dy \\ &=^{(x\leftrightarrow y)} \iint \omega_\varepsilon(x,t)\omega_\varepsilon(y,t)\mathbf{K}(y-x)\cdot y\ dy\ dx \\ &= -\iint \omega_\varepsilon(x,t)\omega_\varepsilon(y,t)\mathbf{K}(x-y)\cdot y\ dy\ dx.\end{aligned}$$

Therefore we have

$$\omega_{\varepsilon,t}(\mathbf{u}_{\varepsilon,t}\cdot X)=\frac{1}{2}\iint \omega_\varepsilon(x,t)\omega_\varepsilon(y,t)(x-y)\cdot\mathbf{K}(x-y)\;dx\;dy=0$$

since by definition $\mathbf{K}(x)\perp x$. So putting this together

$$\frac{d\omega_{\varepsilon,t}}{dt}((X-m_\varepsilon(t))^2)=2\;\omega_{\varepsilon,t}(\mathbf{F}\cdot(X-m_\varepsilon(t)). \tag{2.9}$$

On the other hand

$$\begin{aligned}&\omega_{\varepsilon,t}\left(\frac{dm_\varepsilon}{dt}\cdot(X-m_\varepsilon(t))\right)=\frac{dm_\varepsilon}{dt}\cdot\omega_{\varepsilon,t}(X-m_\varepsilon(t))\\ &\text{and }\omega_{\varepsilon,t}(X-m_\varepsilon(t))=\int\omega_\varepsilon(x,t)(x-m_\varepsilon(t))dx\\ &\qquad\qquad=\int\omega_\varepsilon(x,t)x\;dx-m_\varepsilon(t)\int\omega_\varepsilon(x,t)\;dx=0\end{aligned} \tag{2.10}$$

since $\int\omega_\varepsilon(x,t)x\;dx=m_\varepsilon(t)$ and as in the earlier remark $\int\omega_\varepsilon(x,t)dx=1$. Finally we have

$$\frac{d}{dt}I_\varepsilon(t)=2\int\omega_\varepsilon(x,t)\;\mathbf{F}(x,t)\cdot(x-m_\varepsilon(t))\;dx. \tag{2.11}$$

Now since $\int dx\;(x-m_\varepsilon(t))\cdot\mathbf{F}(m_\varepsilon(t),t)\omega_\varepsilon(x,t)=0$ by the same argument as in (2.10), we can use the Lipschitz continuity of $\mathbf{F}$:

$$\begin{aligned}\left|\frac{d}{dt}I_\varepsilon(t)\right|&=2\left|\int dx\;\omega_\varepsilon(x,t)\;\;(\mathbf{F}(x,t)-\mathbf{F}(m_\varepsilon(t),t))\cdot(x-m_\varepsilon(t))\right|\\ &\le 2L\int dx\;(x-m_\varepsilon(t))^2\omega_\varepsilon(x,t)=2LI_\varepsilon(t).\end{aligned} \tag{2.12}$$

We can apply Grönwall's Inequality and get the bound

$$I_\varepsilon(t)\le I_\varepsilon(0)\exp(2Lt). \tag{2.13}$$

Finally, as at $t=0$, for $\varepsilon\to 0$

$$\begin{cases}\omega_{\varepsilon,0}\rightharpoonup\delta_{x_0}\\ m_\varepsilon(0)\to x_0,\end{cases}$$

$I_\varepsilon(0)\to(x_0-m_\varepsilon(0))^2=0$ so that $\lim_{\varepsilon\to0}I_\varepsilon(t)=0$ uniformly on $t\in[0,T]$.

Step 2. Let us prove (i) and (ii). We have, starting from $\frac{dm(t)}{dt}=\mathbf{F}(m(t),t)$ and $\frac{dm_\varepsilon(t)}{dt}=\omega_{\varepsilon,t}(\mathbf{F})$,

$$\begin{aligned}&|m(t)-m_\varepsilon(t)|\\ &\quad\le|x_0-m_\varepsilon(0)|+\int_0^t ds\;\;|\mathbf{F}(m(s),s)-\omega_{\varepsilon,s}(\mathbf{F})|\\ &\quad\le|x_0-m_\varepsilon(0)|+\int_0^t ds\;\;|\mathbf{F}(m(s),s)-\mathbf{F}(m_\varepsilon(s),s)|+\int_0^t ds\;\;|\mathbf{F}(m_\varepsilon(s),s)-\omega_{\varepsilon,s}(\mathbf{F})|\end{aligned}$$

using $\omega_{\varepsilon,t}(\mathbf{F}(m_\varepsilon(s),s)) = \mathbf{F}(m_\varepsilon(s),s)\ \omega_{\varepsilon,t}(1) = \mathbf{F}(m_\varepsilon(s),s)$ we get

$$\begin{aligned}
&\leq |x_0 - m_\varepsilon(0)| + L\int_0^t ds\ \ |m(s) - m_\varepsilon(s)| + \int_0^t ds\ \ |\omega_{\varepsilon,s}(\mathbf{F}(m_\varepsilon(s),s) - \mathbf{F}(\cdot,s))| \\
&\leq |x_0 - m_\varepsilon(0)| + L\int_0^t ds\ \ |m(s) - m_\varepsilon(s)| + L\int_0^t ds\ \omega_{\varepsilon,s}(|m_\varepsilon(s) - X|)
\end{aligned}$$

Now we use that for a finite measure (as is $\omega_{\varepsilon,t}$),

$$\int \omega_\varepsilon(x,t)\,|m_\varepsilon(s) - x|\,dx \leq \int \omega_\varepsilon(x,t)(m_\varepsilon(s) - x)^2 dx = I_\varepsilon(s).$$

We obtain

$$|m(t) - m_\varepsilon(t)| \leq \underbrace{|x_0 - m_\varepsilon(0)|}_{o(1)_{\varepsilon\to 0}} + L\int_0^t ds\ \ |m(s) - m_\varepsilon(s)| + \underbrace{LT\sup_{s\in[0,T]} I_\varepsilon(s)}_{o(1)_{\varepsilon\to 0}}$$

So that again $|m(t) - m_\varepsilon(t)| \leq C(\varepsilon)\,|x_0 - m_\varepsilon(0)|\exp(Lt)$ (where $C(\varepsilon)\to 0$) and we can conclude $m_\varepsilon(t)\to m(t)$ for the same reasons as before.

On the other hand, if $f \in C_0^\infty(\mathbb{R}^2\to\mathbb{R})$

$$\begin{aligned}
|\omega_{\varepsilon,t}(f) - f(m(t))| &= \left|\int \omega_\varepsilon(x,t)f(x)\ dx - \int f(m(t))\omega_\varepsilon(x,t)\ dx\right| \\
&\leq \int \omega_\varepsilon(x,t)\,|f(x) - f(m(t))|\ dx \leq L'\int \omega_\varepsilon(x,t)\,|x - m(t)|\ dx \\
&\leq L'\left[\int \omega_\varepsilon(x,t)\,|x - m_\varepsilon(t)|\ dx + \int \omega_\varepsilon(x,t)\,|m_\varepsilon(t) - m(t)|\ dx\right] \\
&\leq L'\sup_{t\in[0,T)} I_\varepsilon(t) + |m_\varepsilon(t) - m(t)| \overset{\varepsilon\to 0}{\to} 0.
\end{aligned}$$

Step 3. Finally we show the localisation property. We start by defining

$$\lambda_{\omega_{\varepsilon,t}}(R) := \int_{B(m_\varepsilon(t),R)^c} dx\ \omega_\varepsilon(x,t) \tag{2.14}$$

which is the amount of vorticity outside $B(m_\varepsilon(t),R)$. We will write $\lambda_{\varepsilon,t}$ remembering that it comes from the measure $\omega_{\varepsilon,t}$. Let us estimate this value in terms of $I_\varepsilon(t)$. In one hand

$$\begin{aligned}
I_\varepsilon(t) &= \int \omega_\varepsilon(x,t)(x - m_\varepsilon(t))^2 dx \\
&\geq \int \omega_\varepsilon(x,t)(x - m_\varepsilon(t))^2 \chi(|x - m_\varepsilon(t)| \geq R)\ dx \\
&\geq R^2\int \omega_\varepsilon(x,t)\chi(|x - m_\varepsilon(t)| \geq R)\ dx = R^2\lambda_{\varepsilon,t}(R)
\end{aligned}$$

But we have also $I_\varepsilon(t) \leq I_\varepsilon(0)\exp(2Lt)$ with

$$\begin{aligned} I_\varepsilon(0) &= \varepsilon^{-2}\int_{\Lambda_\varepsilon}\left(x - \varepsilon^{-2}\int_{\Lambda_\varepsilon} y\ dy\right)^2 dx = \varepsilon^{-4}\int_{\Lambda_\varepsilon}\int_{\Lambda_\varepsilon}(x-y)^2\ dx\ dy \\ &\leq (2\alpha\varepsilon)^2 \end{aligned}$$

So that finally the amount of vorticity outside any disc of radius R centered at the center of vorticity goes to 0 on $[0,T]$ as $\varepsilon \to 0$. Indeed,

$$\begin{aligned} \lambda_{\varepsilon,t}(R) &\leq \frac{I_\varepsilon(t)}{R^2} \leq \frac{1}{R^2}\exp(2LT)(2\alpha\varepsilon)^2 \\ &\leq C\frac{\varepsilon^2}{R^2} \overset{\varepsilon\to 0}{\to} 0 \quad \forall\ R > 0, \end{aligned} \tag{2.15}$$

for a constant C depending on T.

Remark : The bound on $I_\varepsilon(t)$ is not enough to conclude. The estimate (2.15) tell us that the contributions of vorticity far from m_ε are negligible (of order at most ε^2). These contributions of the vorticity could produce a non-vanishing field $\forall\ \varepsilon$. We need a more sophisticated analysis to show that $\lambda_{\varepsilon,t}(R) = 0$ for $\varepsilon > 0$ small enough, whatever the $R > d$.

Let decompose the velocity field at a point x in several parts:

$\mathbf{u}_1$ the field generated by the vorticity inside a small disc $B_1 = B(m_\varepsilon(t), R_\varepsilon)$, with $R_\varepsilon \to 0$ as $\varepsilon \to 0$ such that at $t = 0$, $\Lambda_\varepsilon(0) \subset B_1$. We would like to bound the contribution of $\mathbf{u}_1$ that could take us far from $m_\varepsilon(t)$, that is $|\mathbf{u}_1 \cdot n|$ where n is unitary and radial in the direction $m_\varepsilon(t) - x$;

$\mathbf{u}_2$ the field generated by the vorticity outside B_1, we will show $|\mathbf{u}_2| \to 0$;

$\mathbf{u}_3$ the external field, but by Lipschitz continuity of $\mathbf{F}$, a particle near $m_\varepsilon(t)$ feels essentially the same external field as the center, so this contribution will not cause the particle to move away from $m_\varepsilon(t)$.

We place ourselves in $B_2 = B(m_\varepsilon(t), R'_\varepsilon) \supset B_1$, with $R'_\varepsilon \to 0$. In such a point only $\mathbf{u}_1$ and $\mathbf{u}_2$ can contribute and spread Λ_ε. First of all, since $n \cdot \mathbf{K}(x - m_\varepsilon(t)) = 0$, we have

$$\begin{aligned} |\mathbf{u}_1(x)\cdot n| &= \left|n \cdot \omega_{\varepsilon,t}\big|_{B_1} \star \mathbf{K}\right| = \left|n \cdot \int_{B_1}\omega_\varepsilon(y,t)\mathbf{K}(x-y)\ dy\right| \\ &\leq n \cdot \int_{B_1}\omega_\varepsilon(y,t)\left|\mathbf{K}(x-y) - \mathbf{K}(x - m_\varepsilon(t))\right|\ dy \leq C\frac{R_\varepsilon}{(R_\varepsilon - R'_\varepsilon)^2} \to 0 \end{aligned} \tag{2.16}$$

This holds if $R'_\varepsilon = R_\varepsilon^\alpha$, $0 < \alpha < 1/2$, then for $R_\varepsilon < 1$ we do have $B_2 \supset B_1$ as we wanted and also $R'_\varepsilon \to 0$.

Next, we can estimate $\mathbf{u}_2$ via the amount of vorticity escaping from B_1 at each time. We introduce

$$\mu_R(t) := 1 - \varepsilon^{-2}\int_{\Lambda_\varepsilon(0)} dx\ W_R(m_\varepsilon(t) - x_\varepsilon(t)), \quad \text{where}$$

W_R is a cut-off function C_0^∞ such that

– $W_R(x) = \begin{cases} 1 & \text{if } |x| \leq R \\ 0 & \text{if } |x| > 2R \end{cases}$

– $|\nabla W_R(x)| \leq C/R$,

– $\left|\nabla W_R(x) - \nabla W_R(x')\right| \leq \dfrac{C}{R^2}\left|x - x'\right|$

– W_R has spherical symmetry so that $\nabla W_R(x) = W_R'(r)\dfrac{x}{|x|},\ r = |x|$.

We also take $x_\varepsilon(t)$, a particle path leaving x at time $t = 0$. We notice that if $\Lambda_\varepsilon(t) \subset B(m_\varepsilon(t), R)$, then $\mu_R(t) = 0$. This means that $\mu_R(t)$ is a kind of analog of $\lambda_{\varepsilon,t}$, measuring the localisation of $\Lambda_\varepsilon(t)$ around $m_\varepsilon(t)$.

Assuming that we have the inequality

$$\mu_{R_\varepsilon}(t) \leq Ce^{\beta}, \quad \beta > 2,\ t \in [0, T) \tag{2.17}$$

with a convenient form of R_ε, the theorem is proved: $\mathbf{u}_2$, which depends on the vorticity escaping $B_1 = B(m_\varepsilon(t), R_\varepsilon)$ measured by μ_{R_ε}, vanishes too for $\varepsilon \to 0$.

The fields that could cause the particles near $m_\varepsilon(t)$ to escape (essentially $\mathbf{u}_1$ and $\mathbf{u}_2$) are arbitrarily small, so that after a time T, $\Lambda_\varepsilon(t)$ can be contained in a fixed disc $B(m_\varepsilon(t), d)$ assuming ε sufficiently small.

In practice writing $\delta_x^2(t) = |x_\varepsilon(t) - m_\varepsilon(t)|^2$ the distance between any particle starting at x and the center, we have

$$\begin{aligned} \left|\frac{d}{dt}\delta_x^2\right| &= \left|2\left(\frac{dx_\varepsilon(t)}{dt} - \frac{dm(t)}{dt}\right)\cdot(x_\varepsilon(t) - m_\varepsilon(t))\right| \\ &= |C\delta_x(t).n \cdot (\mathbf{u}_1(x_\varepsilon) + \mathbf{u}_2(x_\varepsilon) + \mathbf{u}_3(x_\varepsilon) - \mathbf{F}(m_\varepsilon)| \\ &\leq C\delta_x(t)\left[|n \cdot \mathbf{u}_1| + |n \cdot \mathbf{u}_2| + |\mathbf{F}(x_\varepsilon) - \mathbf{F}(m_\varepsilon)|\right] \end{aligned} \tag{2.18}$$

recalling $\mathbf{u}_3 = \mathbf{F}$, $|n \cdot (\mathbf{F}(x_\varepsilon) - \mathbf{F}(m_\varepsilon))| \leq |\mathbf{F}(x_\varepsilon) - \mathbf{F}(m_\varepsilon)|$. With x starting in $\Lambda_\varepsilon(0)$, so that $\delta_x(0) < 2\alpha\varepsilon$, we have

$$\begin{cases} |n \cdot \mathbf{u}_1| \leq A_1(\varepsilon) \to 0 \\ |n \cdot \mathbf{u}_2| \leq |\mathbf{u}_2| \to 0 \\ |\mathbf{F}(x_\varepsilon) - \mathbf{F}(m_\varepsilon)| \leq L\delta_x(t) \end{cases}$$

Then $\frac{d\delta_x(t)}{dt} \leq C(\varepsilon) + C'\delta_x(t)$, using $\frac{d}{dt}\delta^2 = 2\delta\frac{d\delta}{dt}$. This implies

$$\delta_x(t) \leq \delta_x(0) + T\ C(\varepsilon) + \int_0^t C'\delta_x(t)dt.$$

Using Grönwall's Inequality we conclude that for some constant $C(\varepsilon)$,

$$\delta_x(t) \leq C(\varepsilon)\exp(C't) \leq C(\varepsilon)\exp(C'T), \quad \forall\ t \in [0, T), \tag{2.19}$$

with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. We can then choose, with $T > 0$ fixed, an ε_0 such that $\forall \varepsilon < \varepsilon_0$, all points starting in Λ_ε stay at a distance at most d of the center at all times, i.e. that $\Lambda_\varepsilon(t) \subset B(m_\varepsilon(t), d)$ $\forall\ t \in [0, T)$

To finish, we now prove (2.17), it is quite technical. Taking the derivative we get

$$\frac{d\mu_R}{dt} = -\varepsilon^{-2}\int_{\Lambda_\varepsilon(0)} dx\ \nabla W_R(m_\varepsilon(t)-x_\varepsilon(t))\cdot\frac{d}{dt}(m_\varepsilon(t)-x_\varepsilon(t)) \tag{2.20}$$

but $\frac{dm_\varepsilon}{dt} = \omega_{\varepsilon,t}(\mathbf{F})$, $\frac{dx_\varepsilon}{dt} = \mathbf{u}_\varepsilon(x_\varepsilon(t),t)+\mathbf{F}(x_\varepsilon(t),t)$ a particle feels the external velocity as well as the one generated by the vorticity. We compute

$$\begin{aligned}\frac{d\mu_R}{dt} &= \varepsilon^{-2}\Big[\int_{\Lambda_\varepsilon(0)} dx\ \nabla W_R(m_\varepsilon(t)-x_\varepsilon(t))\cdot\int dy\ \omega_\varepsilon(y,t)\mathbf{K}(x_\varepsilon(t)-y)\\ &\qquad + \int_{\Lambda_\varepsilon(0)} dx\ \nabla W_R(m_\varepsilon(t)-x_\varepsilon(t))\cdot\int dy\ \omega_\varepsilon(y,t)\left(\mathbf{F}(x_\varepsilon(t),t)-\mathbf{F}(y,t)\right)\Big]\\ &:= I_1+I_2\end{aligned} \tag{2.21}$$

We should now estimate I_1 and I_2. By Liouville's Theorem, we know that integrating the variable $x_\varepsilon(t)$ (starting in x at $t=0$) with the measure $\omega_{\varepsilon,0}dx$ is equivalent to integrate x with the measure $\omega_{\varepsilon,t}dx$. So we get

$$\begin{aligned}I_1 &= \iint dx\ dy\ \omega_\varepsilon(x,t)\omega_\varepsilon(y,t)\nabla W_R(m_\varepsilon(t)-x)\cdot\mathbf{K}(x-y)\\ &= \frac{1}{2}\iint dx\ dy\ \omega_\varepsilon(x,t)\omega_\varepsilon(y,t)\left(\nabla W_R(m_\varepsilon(t)-x)-\nabla W_R(m_\varepsilon(t)-y)\right)\cdot\mathbf{K}(x-y)\end{aligned} \tag{2.22}$$

by anti-symmetry, and for I_2 we have

$$I_2 = \iint dx\ dy\ \omega_\varepsilon(x,t)\omega_\varepsilon(y,t)\nabla W_R(m_\varepsilon(t)-x(t))\cdot(\mathbf{F}(x,t)-\mathbf{F}(y,t))\,. \tag{2.23}$$

Starting with I_1, first of all by the definition of ∇W_R, the integral has non-zero contributions only if at least one of x or y are in the annulus $R<|x|<2R$. Let us separate I_1 on various domains, firs on D_1 defined as $x\in B(m_\varepsilon(t),R)^c,\ y\in B(m_\varepsilon(t),\gamma)\subset B(m_\varepsilon(t),R)$ with $0<\gamma<R$.

$$\left|I_1\Big|_{D_1}\right| \le \iint_{D_1} dx\ dy\ \omega_\varepsilon(x,t)\omega_\varepsilon(y,t)\underbrace{|\nabla W_R(m_\varepsilon(t)-x)|}_{\le C/R}\ |\mathbf{K}(x-y)-\mathbf{K}(x-m_\varepsilon(t))| \tag{2.24}$$

using $\nabla W_R(x)\perp\mathbf{K}(x)$, on D_1, $x\neq y, x\neq m_e(t)$ so that $\mathbf{K}$ is C^∞ and Lipschitz in particular

$$\begin{aligned}&\le \iint_{D_1} dx\ dy\ \omega_\varepsilon(x,t)\omega_\varepsilon(y,t)\frac{C_1}{R(R-\gamma)^2}\,|m_\varepsilon(t)-y|\\ &\le \frac{C_1\ \gamma}{R(R-\gamma)^2}\int_{B(m_\varepsilon(t),R)^c} dx\ \omega_\varepsilon(x,t) = C_2\frac{\gamma}{R^3}\lambda_{\varepsilon,t}(R)\end{aligned} \tag{2.25}$$

if $R-\gamma\ge R/2$, i.e. for a sufficiently small γ. Taking for example $\gamma=R^6$ for $R<1$ we finally have

$$\left|I_1\Big|_{D_1}\right| \le C_2R^3\lambda_{\varepsilon,t}(R). \tag{2.26}$$

Now on the domain D_2 defined as $x \in B(m_\varepsilon(t), R)^c,\ y \in B(m_\varepsilon(t), \gamma)^c$, we have by definition that

$$\begin{aligned} &|\mathbf{K}(x-y)| \leq \frac{C}{|x-y|}, \\ &|\nabla W_R(m_\varepsilon(t)-x) - \nabla W_R(m_\varepsilon(t)-y)| \leq \frac{C'}{R^2}\,|y-x|\,. \end{aligned} \tag{2.27}$$

Therefore, we get

$$\begin{aligned} \left|I_1\Big|_{D_2}\right| &\leq C_3 \iint_{D_2} dx\ dy\ \omega_\varepsilon(x,t)\omega_\varepsilon(y,t)\frac{|y-x|}{R^2}\ \frac{1}{|x-y|} \leq \frac{C_3}{R^2}\lambda_{\varepsilon,t}(R)\lambda_{\varepsilon,t}(\gamma) \\ &\leq \frac{C_4}{R^2\gamma^2}\varepsilon^2\lambda_{\varepsilon,t}(R). \end{aligned} \tag{2.28}$$

Finally on domains D_3, D_4 defined in the same way reversing the role of x and y we end up with the same conclusions and using (2.15), we observe that

$$|I_1| \leq C_5\ \varepsilon^2(R + \varepsilon^2 R^{-16}). \tag{2.29}$$

We now consider I_2. We have $|\mathbf{F}(x,t) - \mathbf{F}(y,t)| \leq 2\,|F|_\infty < \infty\ \forall\ x, y$ because $\mathbf{F}$ is bounded, so that if we decompose the domain of the integral in y in $B(m_\varepsilon(t), R)$ and its complement, we get

$$|I_2| \leq C_6\ \lambda_{\varepsilon,t}(R)(1 + \frac{\lambda_{\varepsilon,t}(R)}{R}). \tag{2.30}$$

Finally we deduce that

$$\left|\frac{d\mu_R}{dt}\right| \leq A(R,\varepsilon) \tag{2.31}$$

where A is a function of R, ε that tends to 0 for ε or $R \to 0$, and $A(R,\varepsilon) \geq C\varepsilon^\beta$ for a $\beta > 2$ if ε, R are small enough. We can then conclude. Indeed, if R is small but big enough so that $\mu_R(0) = 0$ ($R > 2\alpha\varepsilon$ works), for example taking $R_\varepsilon = \varepsilon^{1/100} > 2\alpha\varepsilon$ for $\varepsilon < 1$, there is a (big) constant such that for $\beta > 2$,

$$|\mu_R(t)| \leq |\mu_R(0)| + \int_0^t A(R,\varepsilon)\ dt \leq C_7\varepsilon^\beta. \tag{2.32}$$

□

Finally we can prove the main theorem.

Theorem 2.2. We define

$$\omega_{\varepsilon,0} = \varepsilon^{-2}\sum_{i=1}^N \gamma_i \chi_{\Lambda_\varepsilon^i} \tag{2.33}$$

with Λ_ε^i a family of disjoint open regions such that $\Lambda_\varepsilon^i \subset B(z_{i,0}, \alpha x)$ for some $\alpha > 0$, $\lambda(\Lambda_\varepsilon^i) = \varepsilon^2$ and $\gamma_i \in \mathbb{R}$ for $i = 1, \dots N$. Then $\forall\ T > 0$, we have

$$\lim_{\varepsilon\to 0} \omega_{\varepsilon,t}(f) = \sum_{i=1}^N \gamma_i f(z_i(t)), \qquad t \in [0,T) \tag{2.34}$$

where we write $\omega_{\varepsilon,t}$ to denote the weak solution to the Euler Equation (1.11) with initial conditions $\omega_{\varepsilon,0}$, and $\{z_i(t)\}$ the solutions to the vortex model (MOT) with intensities $\{\gamma_i\}$ and initial conditions $\{z_{i,0}\}$, provided that these trajectories do not collide at a finite time $T < +\infty$.

Proof. It is quite simple with the preceding theorem. One can consider each patch of vorticity individually, and if these patches are far enough, the action of the others acts as an external Lipschitz field.

Let $d > 0$ be the minimal distance reached between any two vortices for $t \in [0, T]$ according to the vortex dynamics. We can regularise the vortex model as we did earlier (Equation (1.34)), and apply the same construction to the Euler Equation. Two concentrations of vorticity will interact via the regular kernel $\mathbf{K}_\eta$, $\eta \ll d$ and each one interacts with itself with the singular kernel $\mathbf{K}$. The kernel $\mathbf{K}_\eta$ being regular, it defines a smooth velocity field and we are able to apply Theorem 2.1 replacing the equations with their regularised form. We will conclude the same thing, i.e. the motion of each patch of vorticity tends in the weak sense to the movement of its center of vorticity in the regular external field. For each individual patch with vorticity $\omega^\eta_{\varepsilon,t,i}$, such that the global $\omega^\eta_{\varepsilon,t} = \sum_i \gamma_i \omega^\eta_{\varepsilon,t,i}$ is solution to the regular Euler Equation, will have $\omega^\eta_{\varepsilon,t,i} \to \delta_{z_i^\eta(t)}$ in the weak sense for $\varepsilon \to 0$ (points (i) and (ii) of Theorem 2.1).

Moreover, thanks to (iii), since the patches stay localised around their center of vorticity, we can choose ε small enough so that the minimal space between two blobs is always greater than $d/2$, but then for η small enough given by d, there will be no η-collapse between patches and the regularised dynamics coincides with the standard one. We finally conclude that

$$\omega_{\varepsilon,t} \overset{\varepsilon\to 0}{\to} \sum_i \gamma_i \delta_{z_i(t)}, \quad t \in [0, T) \tag{2.35}$$

which is what we wanted. □

Chapter 2

Applications on domains

2.1 Exact Solutions

We will start our exploration of the applications of the Equation (MOT) by looking at exact solutions in the plane $\mathbb{R}^2$. We will see that for two or three bodies the system is integrable, and explicit solutions can always be found. We will also see which initial conditions lead to a stationary solutions. On the other side, if $N \geq 4$ the dynamic can become chaotic, even though there still are some initial configuration that give static solutions. These discussions are found in Miot (2015).

2.1.1 Two-bodies motion

In the case of two point vortices in the plane, the equations of movement are

$$\begin{aligned} \dot{z}_1 &= \frac{\gamma_2}{2\pi} \frac{(z_1(t) - z_2(t))^\perp}{|z_1(t) - z_2(t)|^2} \\ \dot{z}_2 &= \frac{\gamma_1}{2\pi} \frac{(z_2(t) - z_1(t))^\perp}{|z_1(t) - z_2(t)|^2}. \end{aligned} \tag{1.1}$$

Moreover, we have $H(t) = -\frac{1}{2\pi}\gamma_1\gamma_2 \log|z_1(t) - z_2(t)| = H(0)$ since H is conserved, so that we know that the distance between the two particles,

$$l_{12}(t) := |z_1(t) - z_2(t)| = l_{12}(0) \tag{1.2}$$

is constant during the evolution of the system. This implies that the particles cannot get close, i.e. there is no collision possible the solution exists for all time. Let us make explicit the possible trajectories.

– If $\Gamma = \gamma_1 + \gamma_2 = 0$, i.e. $\gamma_2 = -\gamma_1$, because $M(t) = \gamma_1(z_1(t) - z_2(t)) = M(0)$ the equations are rewritten

$$\dot{z}_1 = \dot{z}_2 = -\frac{\gamma_1}{2\pi} \frac{(z_1(t) - z_2(t))^\perp}{|z_1(t) - z_2(t)|^2} = -\frac{M(0)^\perp}{2\pi\ l_{12}^2} \tag{1.3}$$

this means that z_1 and z_2 move at constant speed, in uniform translation perpendicularly to the segment joining them. And their speed is proportional to the inverse of the distance

between them ($|M| \propto \gamma\ l_{12}$):

$$\begin{aligned} z_1(t) &= z_{1,0} - \frac{M(0)^\perp}{2\pi\ l_{12}^2}\ t \\ z_2(t) &= z_{2,0} - \frac{M(0)^\perp}{2\pi\ l_{12}^2}\ t \end{aligned} \tag{1.4}$$

It is the simplest example of motion of point vortices that is not bounded in the plane.

– If $\Gamma \neq 0$ then the center of vorticity, also called center of inertia in analogy to the standard problems of analytical mechanics involving masses, can be defined by

$$m(t) = \frac{M(t)}{\Gamma} = \frac{\gamma_1}{\gamma_1+\gamma_2} z_1(t) + \frac{\gamma_2}{\gamma_1+\gamma_2} z_2(t) = m(0). \tag{1.5}$$

We can express $\gamma_2(z_1 - z_2) = (\gamma_1+\gamma_2)z_1 - \gamma_1 z_1 - \gamma_2 z_2 = \Gamma z_1 - \Gamma m(t)$ and get

$$\begin{aligned} \dot{z}_1 &= \frac{\Gamma}{2\pi} \frac{(z_1 - m(0))^\perp}{l_{12}^2} \\ \dot{z}_2 &= \frac{\Gamma}{2\pi} \frac{(z_2 - m(0))^\perp}{l_{12}^2} \end{aligned} \tag{1.6}$$

This is the motion of a couple of rotating particle around their center of vorticity and at constant angular velocity $\frac{\Gamma}{2\pi l_{12}^2}$.

These are all the possible outcomes.

2.1.2 Three-bodies motion

The motion of three point vortices in the plane is a bit more interesting, we define

$$\begin{aligned} l_{12}(t) &:= |z_1(t) - z_2(t)|\,, \\ l_{23}(t) &:= |z_2(t) - z_3(t)|\,, \\ l_{31}(t) &:= |z_3(t) - z_1(t)|\,. \end{aligned} \tag{1.7}$$

Expressed in terms of these quantities we get for example

$$\begin{aligned} \frac{d}{dt} l_{12}^2 &= 2(z_1 - z_2)\cdot(\dot{z}_1 - \dot{z}_2) \\ &= \frac{2}{2\pi}(z_1 - z_2)\cdot\left[\gamma_2 \frac{(z_1-z_2)^\perp}{l_{12}^2} + \gamma_3 \frac{(z_1-z_3)^\perp}{l_3 1^2} - \gamma_1 \frac{(z_2-z_1)^\perp}{l_1 2^2} - \gamma_3 \frac{(z_2-z_3)^\perp}{l_2 3^2}\right] \\ &= \frac{\gamma_3}{\pi}\left[(z_1-z_2)\cdot(z_1-z_3)^\perp - (z_1-z_2)\cdot(z_2-z_3)^\perp\right] \end{aligned} \tag{1.8}$$

And we can convince ourselves that $A := (z_2 - z_1)\cdot(z_1 - z_3)^\perp = (z_2 - z_1)\cdot(z_2 - z_3)^\perp$, by cycling trough we have the following equations for the distances between points:

$$\begin{aligned} \frac{d}{dt} l_{12}^2 &= 2A(t)\frac{\gamma_3}{\pi}\left(\frac{1}{l_{23}^2(t)} - \frac{1}{l_{31}^2(t)}\right) \\ \frac{d}{dt} l_{23}^2 &= 2A(t)\frac{\gamma_1}{\pi}\left(\frac{1}{l_{31}^2(t)} - \frac{1}{l_{12}^2(t)}\right) \\ \frac{d}{dt} l_{31}^2 &= 2A(t)\frac{\gamma_2}{\pi}\left(\frac{1}{l_{12}^2(t)} - \frac{1}{l_{23}^2(t)}\right) \end{aligned} \tag{1.9}$$

Coming back to Example 1.9 we see that what allowed us to derive that the ratios of lengths stayed the same is the fact that $\gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1 = 0$ (called h) and

$$\gamma_1\gamma_2 l_{12}^2 + \gamma_2\gamma_3 l_{23}^2 + \gamma_3\gamma_1 l_{31}^2 = 0 = T(t)/2. \tag{1.10}$$

So in general by following exactly the same steps we prove the following theorem:

Theorem 1.1. Assuming three point vortices have initial configuration satisfying $h = 0$, $T(0) = 0$. Then the solution of the motion equation evolves as a self-similar triangle, i.e. with constant ratios of the distances

$$\frac{l_{12}(t)}{l_{23}(t)} = \frac{l_{12}(0)}{l_{23}(0)} \qquad \frac{l_{23}(t)}{l_{31}(t)} = \frac{l_{23}(0)}{l_{31}(0)} \qquad \frac{l_{31}(t)}{l_{12}(t)} = \frac{l_{31}(0)}{l_{12}(0)} \qquad \forall\ t \in [0, T).$$

Moreover, if the tree points do not form an equilateral triangle, nor are aligned at $t = 0$, then $\exists\ C^* \neq 0$ such that

- If $C^* < 0$ the system will collide in an self-similar way,
- if $C^* > 0$ the triangle will grow and the solution is defined for all times.

As we already said earlier, Miot (2015, p.43) adds that all three-bodies systems that will lead to a collision in finite time are necessarily self-similar.

We are now interested in what initial condition are at equilibrium.

Definition 1.2 (Equilibria). We say that a configuration of vortex points is at *equilibrium* if

i. $z_j(t) = z_j(0)$, $\forall\ j = 1, \dots N$, $\forall\ t \in [0, T)$, or

ii. $\exists\ z_0 \in \mathbb{R}^2$ such that $\forall\ t \in [0, T)$, there is a $\omega(t) \in \mathbb{R}, \omega \neq 0$ such that

$$\dot{z}_j(t) = \omega(t)(z_j(t) - z_0)^{\perp}, \quad \forall\ j = 1, \dots N.$$

When (i) holds we call the solution a *stationary* configuration, and when (ii) holds, the system of particles is in rotation around the center z_0, but keeps its shape and size, we call this solution a *relative* equilibrium.

In the case of three points, the first step to get an equilibrium is to ask that the lengths between points remain constant. This is obtained from Equation (1.9) by setting the particles at the vertices of an equilateral triangle, or along a line. Then to be sure that the shape they are in (the triangle which stays the same) does not have itself a non-stable motion, we have to add conditions on the intensities and their relation to the lengths involved.

We get a relative equilibrium for three points at the vertices of an equilateral triangle of side length l with any intensities such that $\Gamma \neq 0$. Then the system will rotate at constant speed Γ/l^2.

For three co-linear points, depending on the intensities we find either a class of initial conditions that will uniformly rotate, or one, two, three or even no equilibrium configuration.

Remark i: Even if the system is integrable for $N = 3$, if we put a fourth 0-vorticity particle in the global velocity field induced by the three vortices, the movement of that particle can become chaotic (in the sense that it is strongly dependent on the initial conditions). This is called the

reduced 4-bodies problem by analogy with the "reduced three-bodies problem" known in celestial mechanics.

Remark ii: For more than three point vortices, generally the system is no longer integrable. There are no analytical solutions that can globally be studied for $t \in [0, T)$, and numerical methods are introduced.

Remark iii: Nevertheless, no matter the number of points that are considered, we can achieve a relative equilibrium putting vortices with same intensities at the ends of a regular N-face polygon. It will then rotate around its center. A broader discussion of this is presented in Miot (2015), and in particular the theory that allows to count all the possible stationary configuration is developed.

2.2 Domains With Boundaries

We come back to Equation (1.4), describing the evolution of a pair of points of opposite intensity. The second point vortex can be seen as the reflection of the first, through the bisector of the segment joining them, such that no velocity crosses that line. The dynamic can then be described completely by a lone vortex, but in the half plane. In this section we will re-derive the properties we already found for this system, but with this new approach. In the same way if we are interested in the motion of a 3D vortex ring, by cylindrical symmetry this is analog to considering a single vortex point in the half plane too.

In fact in our derivation of the equation of motion we used tools that can be generalised to other spaces than the whole Euclidean plane. Here we will see what happens if we constrain the particles to move inside bounded regions. We start by observing the implications of the boundary conditions required to limit the movement, in the general case. Then we will apply a well known tool in physics, the method of images, to study various examples in some notable domains with symmetries. The general method and most of the examples come from Newton (2001), while Example 2.2, Example 2.4 can also be found in Marchioro and Pulvirenti (1994) and Example 2.5 comes from Vasconcelos, Moura, and Schakel (2011).

2.2.1 General theory

We had the equations

$$\frac{d}{dt}\omega(f) = \omega(\mathbf{u}\cdot\nabla f) \qquad \mathbf{K} = \nabla^{\perp} G_D \qquad \mathbf{u}(z) = \mathbf{K}\star\omega(z) = \nabla^{\perp}\left(\sum_{i=1}^{N}\gamma_i G_D(z;z_i)\right) \tag{2.1}$$

where $G_D(x;y)$ id the Green function on the domain $D \subset \mathbb{R}^2$. If the domain now has boundaries, the expression of G is going to change.

Definition 2.1. A *domain* with solid boundary is a subset $D \subset \mathbb{R}^2$. We will always suppose that its boundary ∂D is regular enough for our purposes, sometimes allowing angles but remaining at least continuous and piecewise differentiable. The fact that the boundary is solid means that it should not be crossed by the fluid, we impose the boundary condition

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \text{on } \partial D \tag{2.2}$$

with $\mathbf{n}$ the normal vector to ∂D the boundary at all times. This means that $\mathbf{u}$ is parallel to ∂D at all times.

The dynamics we described for one, two, or three vortices changes a lot depending on the actual shape of D. It could be closed, simply or multiply connected, or even unbounded but containing a solid boundary. The main reason for that change in behaviour is the fact that the symmetry argument we used to convince ourselves that a single vortex in the plane should not move does not hold anymore. In general, a single point vortex in a domain with boundaries starts to move. Moreover, the constants of motion we had came directly from the infinitesimal invariance laws, according to Nöether's theorem. Adding constraints such as (2.2) reduces the number of conservation laws, and we expect less integrability of the system. The number of points allowing us to integrate the system is smaller.

We remember that introducing the stream function ψ, we had $u = \nabla^{\perp}\psi$ and so the no-penetration condition comes to impose that ψ is constant along the boundary. Since ψ is always defined up to an additive constant, we often ask

$$\begin{cases} \Delta\psi = \omega & z \in D, \\ \psi(z) = 0 & z \in \partial D. \end{cases} \tag{2.3}$$

To find the solution of this Dirichlet problem, one must know the form of the Green function on D. It is the (weak) solution to the couple of equations

$$\begin{aligned} &\Delta G(z; z_\alpha) - \delta_{z_\alpha}(z) = 0 \quad && z \in D, \\ &G(z; z_\alpha) = 0 && z \in \partial D. \end{aligned} \tag{2.4}$$

where δ_{z_α} denotes the Dirac measure centered at the singular source $z_\alpha \in D$. Since D is still a subset of $\mathbb{R}^2$, we use the known fundamental solution of the Laplacian on the plane, $G_{\mathbb{R}^2}(z; z_\alpha) = \frac{1}{2\pi}\log|z - z_\alpha|$. Then the Green function can be rewritten

$$G(z; z_\alpha) = G_{\mathbb{R}^2}(z; z_\alpha) + G_D^{(\alpha)}(z) \tag{2.5}$$

for some harmonic ($\Delta G_D^{(\alpha)} = 0$) function $z \mapsto G_D^{(\alpha)}(z)$ at least C^2. The whole problem is now to solve

$$\begin{aligned} &\Delta G_D^{(\alpha)}(z) = 0 && z \in D, \\ &G_D^{(\alpha)}(z) = -G_{\mathbb{R}^2}(z; z_\alpha) = -\frac{1}{2\pi}\log|z - z_\alpha| \quad && z \in \partial D. \end{aligned} \tag{2.6}$$

The function $G_D^{(\alpha)}$ will be related at the same time on the shape of the chosen domain D and on the location of the source z_α. In facts if the domain has no symmetry it is very complex to find the explicit form of a function satisfying those properties.

Remark : Assuming it exists, is this harmonic function unique? In a closed bounded set, the answer is yes. By considering two functions $G_{(1)}, G_{(2)}$, then $w := G_{(1)} - G_{(2)}$ must satisfy the homogeneous equation

$$\begin{cases} \Delta w = 0 & D, \\ w = 0 & \partial D. \end{cases} \tag{2.7}$$

And we can conclude, because on such domain the only solution to this problem is $w = 0$, i.e. $G_{(1)} = G_{(2)}$.

On the other hand, if D is not bounded (on the half-plane for example) we have to add a condition to the form of G_D at infinity to have uniqueness. Otherwise we can define

$$\begin{cases} G_1 = y \\ G_2 = xy \end{cases} \tag{2.8}$$

we get for both functions $\Delta G_i = 0$ and $G_i = 0$ on $y = 0$ but $G_1 \neq G_2$.

For a system of N point vortices, the global Green function will be

$$G^{(N)}(z; z_1, \dots z_N) = \sum_{i=1}^{N} G(z; z_i) = \sum_{i=1}^{N} G_{\mathbb{R}^2}(z; z_i) + \sum_{i=1}^{N} G_D^{(i)}(z), \tag{2.9}$$

which is the sum of the contributions of all vortices. The motion of a vortex point $z_i(t)$ will be influenced by the velocity field at that point, namely

$$\begin{aligned} \mathbf{u}(z,t) = \nabla^\perp \psi &= \sum_j \gamma_j \nabla^\perp G(z; z_j(t)) \\ &= \sum_j \gamma_j \nabla^\perp G_{\mathbb{R}^2}(z; z_j(t)) + \sum_j \gamma_j \nabla^\perp G_D^{(j)}(z) \\ &= \sum_j \gamma_j \mathbf{K}_{\mathbb{R}^2}(z; z_j(t)) + \sum_j \gamma_j \nabla^\perp G_D^{(j)}(z) \end{aligned} \tag{2.10}$$

and as earlier we remove the self-interacting term $\mathbf{K}_{\mathbb{R}^2}(z_i, z_i)$ which is singular. This yields

$$\frac{dz_i(t)}{dt} = \mathbf{u}(z_i(t), t) = \sum_{j \neq i} \gamma_j \mathbf{K}_{\mathbb{R}^2}(z_i(t); z_j(t)) + \sum_j \gamma_j \nabla^\perp G_D^{(j)}(z_i(t)) \tag{2.11}$$

Remark i: G_D is regular everywhere inside the domain, so that $\Delta G_D = 0$. Thus we shall not remove the term $G_D^{(i)}(z_i)$, it is this term that will cause a single point to move. For $N = 1$, we now have

$$\frac{dz_1(t)}{dt} = \gamma_1 \nabla^\perp (G_D^{(1)}(z_1(t))) \tag{2.12}$$

and, as announced, the particle starts to move in the presence of walls.

Remark ii: By the symmetry of the Green function ($G(z; z_\alpha) = G(z_\alpha; z)$) a direct consequence is that $G_D^{(\alpha)}(z_\beta) = G_D^{(\beta)}(z_\alpha)$.

The system is still Hamiltonian and so we define

$$H(z_1, \dots z_N) = -\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \gamma_i \gamma_j G_{\mathbb{R}^2}(z_i; z_j) - \frac{1}{2} \sum_{i,j=1}^{N} \gamma_i \gamma_j G_D^{(j)}(z_i) \tag{2.13}$$

we can then verify that

$$\begin{aligned}\gamma_i \dot{z}_i^1 &= \frac{\partial H}{\partial z_i^2},\\ \gamma_i \dot{z}_i^2 &= -\frac{\partial H}{\partial z_i^1}.\end{aligned} \tag{2.14}$$

In the case of a single vortex, we write instead

$$H = -\frac{1}{2}\gamma_1 G_D^{(1)}(z_1)$$
$$\text{so that } \begin{cases}\dot{z}_1^1 = \dfrac{\partial H}{\partial z_1^2}\\ \dot{z}_1^2 = -\dfrac{\partial H}{\partial z_1^1}\end{cases} \tag{2.15}$$

and the vortex trajectory are given by $H = \text{Cst}$. We call this function H the *Kirchhoff-Routh path* function.

While the initial problem only had singularities when two point vortices meet, here another problematic situation is when the particles reach the boundary. Thankfully we have a theorem, from Flutcher and Gustaffson (see Newton, 2001, p.133), which says that such collisions do not happen unless at least two vortices also collide in the interior of the domain.

2.2.2 Method of images

As we said, it is in general a very difficult problem to find G_D explicitly. The method of images, already known and used in other parts of physics, consists of exploiting the symmetries of the domain to eliminate G on the boundary. To do so, we place virtual vortices outside the domain in positions and with intensities appropriately chosen; we use the fact that if the singularity is placed outside D, at $z_\alpha \notin D$, then

$$G_D^{(\alpha)}(z) = G_{\mathbb{R}^2}(z; z_\alpha) = \frac{1}{2\pi}\log|z - z_\alpha| \tag{2.16}$$

is harmonic and has a similar behaviour to $G_{\mathbb{R}^2}(z; z_i)$ on the boundary. It is easier to understand with examples.

Example 2.2 (Half-plane). Let $D = \{z = (x, y) \in \mathbb{R}^2 \mid y > 0\}$

$$\partial D = \{z = (x, y) \mid y = 0\}$$

With a single point vortex $z_1(t) = (x_1(t), y_1(t))$ of intensity γ, we decompose G into the sum

$$\begin{aligned}G(z) &= G_{\mathbb{R}^2}(z; z_1) + G_D^{(1)}(z)\\ &= \frac{1}{2\pi}\log|z - z_1| - \frac{1}{2\pi}\log|z - z_1^*| \qquad \text{with } z_1^* = (x_1, -y_1)\end{aligned} \tag{2.17}$$

The second term G_D is harmonic in the upper half-plane and by symmetry, on ∂D, $G_{\mathbb{R}^2} = -G_D^{(1)}$.

The stream function becomes

$$\psi(z) = \frac{\gamma}{2\pi}\log|z - z_1| - \frac{\gamma}{2\pi}\log|z - z_1^*| = \psi_1 + \psi_2. \tag{2.18}$$

Here ψ_1 is the contribution of the interaction between the vortex $z_1(t)$ of vorticity γ and the particle at z, and ψ_2 has the form of an interaction between z and a vortex placed at $z_1^*(t)$ with vorticity $-\gamma$ that does not actually exist, i.e. the reflection of z_1 through the vertical line. It formalises what we said, that two points of opposed intensities in the whole plane are equivalent to a single point vortex in the half-plane.

The Hamiltonian is $H(z_1) = -\frac{\gamma}{2}G_D^{(1)}(z_1) = \frac{\gamma}{4\pi}\log|z_1 - z_1*| = \frac{\gamma}{4\pi}\log|2y_1|$ so that a single vortex moves according to

$$\begin{pmatrix}\dot{x}_1\\ \dot{y}_1\end{pmatrix} = \begin{pmatrix}\dfrac{\partial H}{\partial y_1}\\ -\dfrac{\partial H}{\partial x_1}\end{pmatrix} = \begin{pmatrix}\frac{\gamma}{4\pi y_1}\\ 0\end{pmatrix}. \tag{2.19}$$

We find the property that we had in Section 2.1.1, with $l_{12} = 2y_1$.

For each vortex particle added inside the domain, we actually add the same number of virtual vortices outside. The two-bodies problem in the half plane then corresponds to the dynamic of four point vortices disposed in symmetric couples. Let $z_1(t), z_2(t)$ be those two vortices, of vorticity respectively γ_1 and γ_2. We add the two reflections z_1^*, z_2^* of vorticities $-\gamma_1, -\gamma_2$, then

$$\begin{aligned} G^{(2)}(z; z_1, z_2) &= G(z; z_1) + G(z; z_2) \\ &= G_{\mathbb{R}^2}(z; z_1) + \underbrace{G_D^{(1)}(z)}_{=-G_{\mathbb{R}^2}(z;z_1^*)} + G_{\mathbb{R}^2}(z; z_2) + \underbrace{G_D^{(2)}(z)}_{=-G_{\mathbb{R}^2}(z;z_2^*)}. \end{aligned} \tag{2.20}$$

Therefore we have

$$\begin{aligned} \psi(z) &= \sum_j \gamma_j G_{\mathbb{R}^2}(z; z_j) + \sum_j \gamma_j G_D^{(j)}(z) \\ &= \underbrace{\frac{\gamma_1}{2\pi}\log|z - z_1|}_{\text{interaction with } z_1} + \underbrace{\frac{\gamma_2}{2\pi}\log|z - z_2|}_{\text{interaction with } z_2} - \underbrace{\frac{\gamma_1}{2\pi}\log|z - z_1^*|}_{\text{interaction with } z_1^*} - \underbrace{\frac{\gamma_2}{2\pi}\log|z - z_2^*|}_{\text{interaction with } z_2^*} \end{aligned} \tag{2.21}$$

$$\begin{aligned} \text{and } H(z_1, z_2) = \frac{1}{2\pi}(&-\gamma_1\gamma_2\log|z_1 - z_2| + \frac{1}{2}\gamma_1^2\log|z_1 - z_1^*| \\ &+ \gamma_1\gamma_2\log|z_1 - z_2^*| + \frac{1}{2}\gamma_2^2\log|z_2 - z_2^*|). \end{aligned} \tag{2.22}$$

In addition to the energy H being conserved, the only symmetry remaining here is the translation in the horizontal direction, so by Nöether's Theorem we have the constant of motion

$$M^1 = \sum_{i=1}^{2}\gamma_i x_i. \tag{2.23}$$

This allows the problem of two vortices to be integrable, and by doing the numerical integration one realises the system can behave qualitatively in several ways. Assume both vertices have a positive intensity $\gamma_1 = \gamma_2 = \gamma$.

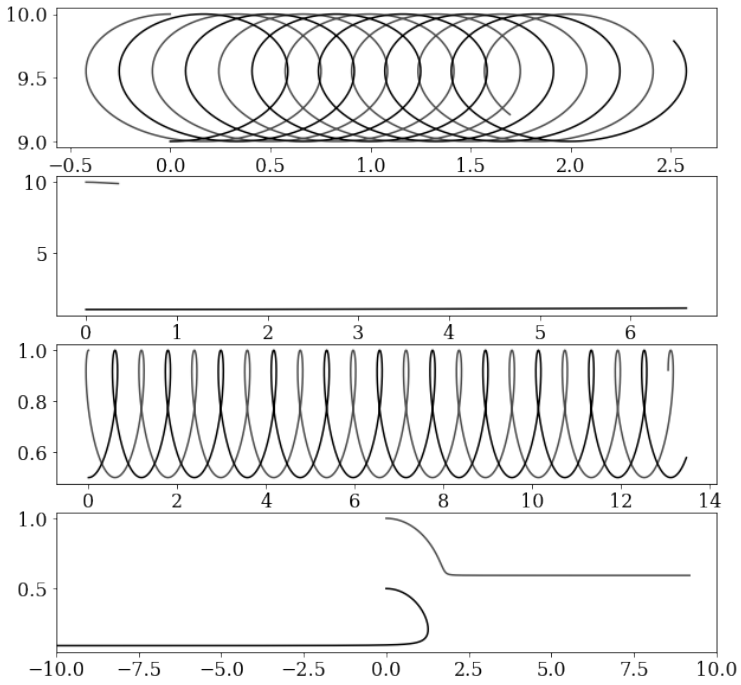


Figure 2.1: The motion of two vortices in several cases. (a) Far from the boundary; (b) One close and one far away; (c) Both close; (d) Opposite intensities.

– If both vortices are far from the boundary, its contribution gets small and we come back nearly in the whole plane case. In a sense the scales of the system do not interact, the two vortices rotate around each other as if there was no wall, but with their center of vorticity moving forward, as a single vortex in the half-plane (Figure 2.1a).

– If one of them is near the boundary and the other far away, because the boundary causes the vortices to move forward at a speed proportional to the inverse of the distance to it, the closest point is carried away very quickly and will no longer interact with the first one (Figure 2.1b).

– If both are close to the boundary, an intermediate situation occurs, the interaction between the particles and the action of the plane have similar intensities (Figure 2.1c).

– If they have opposite vorticity, the plane makes them go respectively to $+\infty$ and $-\infty$. After some interaction, they each go their way (Figure 2.1d).

Adding more point vortices the problem becomes more and more complex. △

In this example we have guessed the expression of G, then realised that it was like adding a point vortex. The general method consists in reversing that problem. The location and intensities of the virtual vortices are guessed by taking advantage of the symmetries. Then we write the stream function $\psi(z)$ as if it was the standard vortex model in the plane (with the real and virtual vortices), knowing that a points interacts with a vortex via the kernel $G_{\mathbb{R}^2}(z; z_\alpha)$. Finally the expression of $G = G_{\mathbb{R}^2} + G_D$ can be deduced, and its validity be checked, and H which will describe the motion of the real vortex points.

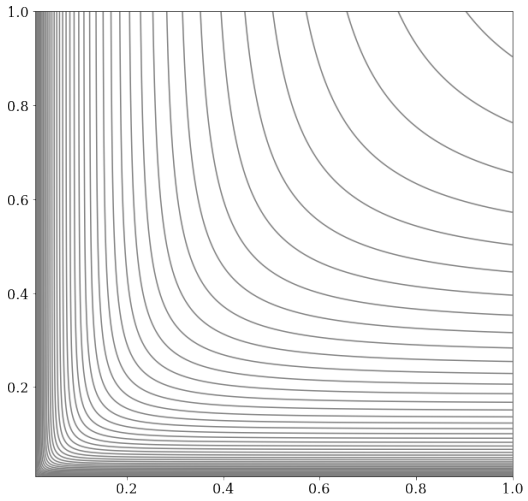


Figure 2.2: Possible trajectories of a vortex in the quadrant.

Example 2.3 (Upper-right Quadrant). Let a vortex $z_1(t) = (x_1(t), y_1(t))$ of intensity γ in the quadrant

$$\begin{aligned} D &= \{z = (x, y) \in \mathbb{R}^2 \mid x, y > 0\} \\ \partial D &= \{x = 0,\ y \geq 0\} \cup \{y = 0,\ x \geq 0\} \end{aligned} \tag{2.24}$$

We place the image points $-\gamma$ at $(x_1, -y_1) = z_1^*$,
$-\gamma$ at $(-x_1, y_1) = -z_1^*$,
γ at $(-x_1, -y_1) = -z_1$.

$$\begin{aligned} \text{Then } \psi(z) &= \psi_{(\gamma, z_1)}(z) + \psi_{(\gamma, -z_1)}(z) + \psi_{(-\gamma, z_1^*)}(z) + \psi_{(-\gamma, -z_1^*)}(z) \\ &= \frac{\gamma}{2\pi} \left(\log|z - z_1| + \log|z + z_1| - \log|z - z_1^*| - \log|z + z_1^*|\right) \\ &= \gamma(G_{\mathbb{R}^2}(z; z_1) + G_D^{(1)}(z)). \end{aligned} \tag{2.25}$$

So

$$G_D^{(1)}(z) = \frac{1}{2\pi} \left(\log|z + z_1| - \log|z - z_1^*| - \log|z + z_1^*|\right) \tag{2.26}$$

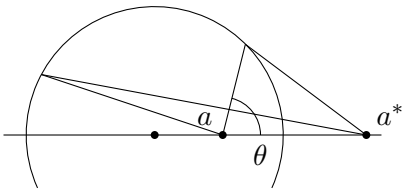
$$\text{and } G(z; z_1) = \frac{1}{2\pi} \left(\log|z - z_1| + \log|z + z_1| - \log|z - z_1^*| - \log|z + z_1^*|\right). \tag{2.27}$$

We can check that indeed $G = 0$ on ∂D. And finally the Hamiltonian

$$\begin{aligned} H(z_1) &= -\frac{1}{2}\gamma G_D^{(1)}(z_1) \\ &= -\frac{\gamma}{4\pi} \left(\log|z + z_1| - \log|z - z_1^*| - \log|z + z_1^*|\right) \end{aligned} \tag{2.28}$$

The motion of a particle is given by the level curves of H (Figure 2.2). With two vortices the problem already becomes much more complicated, as if there were 8 particles interacting in the plane. △

Example 2.4 (Inside a circular domain). We place a vortex inside a circle of radius R, at the position $z_1(t) = (x_1(t), y_1(t))$ with intensity γ.



Using the symmetry of the system, we can limit ourselves to study the case where $z_1 = (a, 0)$. By symmetry once again we know that a virtual vortex would have to be placed outside the boundary, on the x-axis and with negative vorticity. We will try to solve this with a vortex of intensity $-\gamma$ at $(a^*, 0)$. Clearly, it is impossible with that only point to get 0 everywhere on the surface of the circle, otherwise $|z - z_1| = |z - z_1^*|$ for all $|z| = R$, which is impossible. However, we could try to find $\psi_1 + \psi_2 = \text{Cst}$ on that boundary; then $\psi_{tot} = \psi_1 + \psi_2 - \text{Cst}$ will be our stream function.

We have to solve

$$\psi_1(z) + \psi_2(z) = \frac{\gamma}{2\pi} \log|z - z_1| - \frac{\gamma}{2\pi} \log|z - z_1^*| = C \qquad \text{if } |z| = R. \tag{2.29}$$

Then G will have the expression

$$\frac{1}{2\pi} \left(\log|z - z_1| - \log|z - z_1^*|\right) = C/\gamma \tag{2.30}$$

$$\Leftrightarrow \quad \frac{|z - z_1|^2}{|z - z_1^*|^2} = C' \qquad \forall \ |z| = R \tag{2.31}$$

But a point z on the boundary of the circle can be written $z = (R\cos\theta, R\sin\theta)$ so that

$$\begin{cases} |z - z_1|^2 &= R^2 - 2R\cos\theta a + a^2 \\ |z - z_1^*|^2 &= R^2 - 2R\cos\theta a^* + a^{*2} \end{cases} \tag{2.32}$$

The condition (2.31) can then be solved, and it is found that $a^* = \frac{R^2}{a}$, $C = \frac{\gamma}{2\pi} \log\left(\frac{a}{R}\right)$ satisfy the conditions stated before.

In general, we have $z_1^* \propto z_1$, with $|z_1^*| = \frac{R^2}{|z_1|}$ so $z_1^* = \frac{R^2}{|z_1|^2} z_1$, and the stream function is

$$\begin{aligned} \psi(z) &= \frac{\gamma}{2\pi} \log|z - z_1| - \frac{\gamma}{2\pi} \log|z - z_1^*| - \frac{\gamma}{2\pi} \log\left(\frac{|z_1|}{R}\right) \\ &= \gamma \ G_{\mathbb{R}^2}(z; z_1) + \gamma \ G_D^{(1)}(z), \end{aligned} \tag{2.33}$$

$$\text{with } G_D^{(1)}(z) = -\frac{1}{2\pi} \log|z - z_1^*| - \frac{1}{2\pi} \log\left(\frac{|z_1|}{R}\right) \tag{2.34}$$

$$H(z_1) = -\frac{\gamma}{2} G_D^{(1)}(z_1) = \frac{\gamma}{4\pi} \left(\log|z_1 - z_1^*| + \log\frac{|z_1|}{R}\right) \qquad z^* = \frac{R^2}{|z|^2} z \tag{2.35}$$

We have found a candidate G_D that solves our problem in the circular domain, by its uniqueness we know it is the only one. We have solved the problem.
We see that a point vortex feels the interaction with the virtual vortex z_1^* outside the boundary, but also the one of a vortex placed at the origin. It moves according to the level curves of H, that can be expressed as $|z_1| = \text{Cst}$, i.e. the circles centered at the origin.

Once we have the expression of $G_D^{(\alpha)}(z)$ for a vortex z_α, we can write the Hamiltonian describing the motion of any number of particles. Then doing the numerical integration we get some pictures of what the movement of vortices inside a circular domain looks like (Figure 2.3).

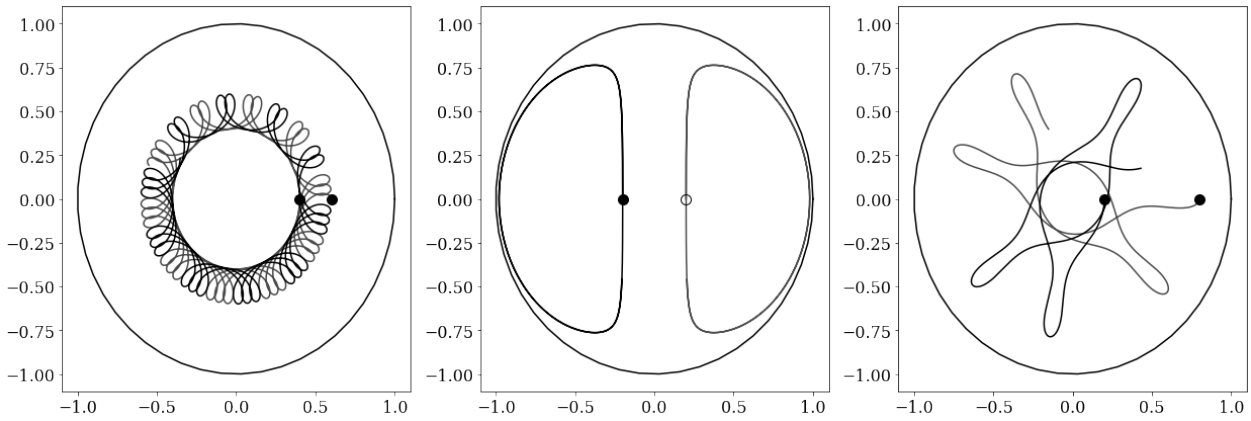


Figure 2.3: Several trajectories of point vortices in the unit circle (whole black dots represent the starting points of positive vorticities and the empty ones are the negative vorticities).

Finally, thanks to the rotational invariance of the system, we also get the constant of motion

$$I = \sum_{i=1}^{N} \gamma_i \left|z_i\right|^2, \tag{2.36}$$

only one left with H. This allows the system of two particles to be integrable. △

Remark i: It seems convenient in these examples to consider that we are in the complex plane ($z(t) = x(t) + i\ y(t),\ \bar{z} = x - iy$) so that the virtual vortices have a simpler form ($z^* = \bar{z}$ in the half plane for example).

We now consider a lone vortex point in the whole plane, at $z_1 = x_1 + iy_1$ and with velocity $\dot{z}_1 = \dot{x}_1 + i\dot{y}_1 := u_1 + iv_1$. By property of the logarithm, we have $\frac{d\log z}{dz} = 1/z = \bar{z}/\left|z\right|^2$ so by defining a complex potential

$$w(z) = i\frac{\gamma}{2\pi}\log(z - z_1), \tag{2.37}$$

$$\text{we have } \frac{dw}{dz} = -u + iv = -\frac{d\bar{z}}{dt} \tag{2.38}$$

for a particle of fluid $z(t)$ with velocity $u(t) + iv(t)$ in a plane with a vortex γ placed at z_1. We could re-do all the work in terms of that complex form but it is not too hard to convince ourselves that everything is going to work. For N vortices $(z_1, \dots z_N)$ of intensities $\gamma_1, \dots \gamma_N$, if we write the complex potential

$$w(z) = \sum_{j=1}^{N} i\frac{\gamma_j}{2\pi}\log(z - z_i), \tag{2.39}$$

then the motion equations are

$$\begin{aligned} \dot{z}_j(t) &= u_j(t) + iv_j(t) \\ \text{with } -u_j(t) + iv_j(t) &= \frac{d\left(\sum_{k\neq j} i\frac{\gamma_k}{2\pi}\log(z - z_k)\right)}{dz}(z_j) =: \frac{d\hat{w}^j}{dz}(z_j) \end{aligned} \tag{2.40}$$

where $\hat{w}^j$ is used to denote that the self-interacting term $z_j \leftrightarrow z_j$ is omitted. Moreover, because $\log z = \log|z| + i\arg z$, it is seen that $\mathfrak{Im}(w) = \psi$ the stream function. We have actually uncovered a very general fact. Writing the system in complex form, one can always define a *complex potential* $w(z = x + iy) = \varphi(x, y) + i\psi(x, y)$. This is a holomorphic function with φ the velocity potential and ψ the stream function, linked by the Cauchy-Riemann Equations satisfied by w:

$$\begin{aligned} \frac{\partial\varphi}{\partial x} &= \frac{\partial\psi}{\partial y} \\ \frac{\partial\varphi}{\partial y} &= -\frac{\partial\psi}{\partial x}. \end{aligned} \tag{2.41}$$

And we have in coordinates form the 2D velocity field $u(x, y) = \nabla^\perp\psi = -\nabla\varphi$ which explains the name *potential* of φ.

Finally in the examples above, the complex potentials are respectively
For the half-plane, $z^* = \bar{z}$, and

$$w(z) = i\frac{\gamma}{2\pi}\log(z - z_1) - i\frac{\gamma}{2\pi}\log(z - \bar{z_1}). \tag{2.42}$$

On the quadrant, $z^* = \bar{z}$ and

$$w(z) = i\frac{\gamma}{2\pi}\log(z - z_1) + \frac{\gamma}{2\pi}\log(z + z_1) - i\frac{\gamma}{2\pi}\log(z - \bar{z_1}) - i\frac{\gamma}{2\pi}\log(z + \bar{z_1}). \tag{2.43}$$

On the circle, $z^* = R^2\frac{z}{|z|^2} = R^2/\bar{z}$ and

$$w(z) = i\frac{\gamma}{2\pi}\log(z - z_1) - i\frac{\gamma}{2\pi}\log(z - R^2/\bar{z_1}) - i\frac{\gamma}{2\pi}\log(z_1/R). \tag{2.44}$$

Remark ii: While already being interesting by itself, the circular domain example is actually very important for the study of vortex motion in any domain with boundaries. Indeed, there are so-called *Conformal mapping* techniques that, given the Hamiltonian of the system in one domain D_0, allow to compute the Hamiltonian in any domain that is the image by some conformal map of D_0. A conformal map is a function of the variables that preserves angles, in the complex plane it is simply a holomorphic function.

Without going into the details (the reader can refer to Newton (2001, Section 3.3) for the derivation), it is found that writing the original Hamiltonian $H_0(x_1, y_1)$ for a single vortex of intensity γ in the domain D_0 (in the complex plane $z_1 = x_1 + iy_1$), and the new domain being mapped via $\zeta(z)$ with Hamiltonian $\hat{H}$:

$$\hat{H}(\zeta(z_1) = \xi_1 + i\eta_1) = H_0(x_1, y_1) + \frac{\gamma}{4\pi}\log\left|\frac{d\zeta}{dz}(z_1)\right|. \tag{2.45}$$

So if we have a closed domain D that is mapped to the unit circle through ζ, with Hamiltonian

$$\hat{H}(\zeta_1) = \frac{\gamma}{4\pi}\left(\log\left|\zeta_1 - 1/\bar{\zeta_1}\right| + \log|\zeta_1|\right) = \frac{\gamma}{4\pi}\log\left|\zeta_1\bar{\zeta_1} - 1\right| \tag{2.46}$$

$$\begin{aligned} \text{then } H_0(z_1) &= \frac{\gamma}{4\pi}\log\left|\zeta(z_1)\bar{\zeta}(z_1) - 1\right| - \frac{\gamma}{4\pi}\log\left|\frac{d\zeta}{dz}(z_1)\right| \\ &= -\frac{\gamma}{4\pi}\log\left(\frac{1}{\left|1 - \zeta_1\bar{\zeta_1}\right|}.\left|\frac{d\zeta}{dz}(z_1)\right|\right). \end{aligned} \tag{2.47}$$

This can then be used to extend the expression of the Hamiltonian in a circular boundary to any closed simply connected bounded domain, that is equivalent to the circle through some conformal mapping.

Example 2.5 (Föppl Equilibrium). Until now, we only considered simply connected, and even convex domains. We will now give an example flow in a multiply connected domain that has great physical and engineering interest. The motion of a fluid around a solid cylinder in the plane, and more specifically the study of the arising so-called Föppl Equilibria for a couple of point vortices in that flow. This example comes from the paper of Vasconcelos et al. (2011)

We place ourselves in the plane, with a solid circular domain and a uniform flow in the x direction. It is realised that the expression of ψ inside a circle works also outside of it. If z_1 is a vortex outside the domain, its symmetric z_1^* will be inside, and we do have $\psi = 0$ on the boundary. This comes from the symmetry between the inside and outside of a circular domain, since $z^{**} = z$. Another way to realise this is by taking the conformal map $z \mapsto 1/z$ that reverses the inside and outside of the unit circle, and use the expression we stated. So if we consider two points vortices z_1, z_2 of intensities respectively γ_1, γ_2 outside the unit circle, the stream function is

$$\psi(z) = \frac{1}{2\pi}\Big[\gamma_1 \log|z - z_1| + \gamma_2 \log|z - z_2| - \gamma_1 \log|z - z_1^*| - \gamma_2 \log|z - z_2^*| \\ - \gamma_1 \log|z_1| - \gamma_2 \log|z_2|\Big]. \quad (2.48)$$

Here we are interested in the behaviour of two initially symmetric point vortices, $\gamma_2 = -\gamma_1 := -\gamma$ and $z_2(0) = \bar{z}_1(0)$. We also want to add the effects of a uniform flow field U in the x direction, it will be disturbed by the presence of the cylinder. To satisfy the solid boundary condition, Milne-Thomson's Circle Theorem (Newton, 2001, p.126) states that if f is a complex potential in the plane, then the total potential describing the fluid motion in presence of a circular boundary of radius a is

$$w(z) = f(z) + f^*(a^2/z), \qquad \text{where } f^*(z) := \bar{f}(\bar{z}) \quad (2.49)$$

And w has the same singularities than f in the domain. This is easily verified, since on $|z| = a$, $\bar{z} = a^2/z$ and so $w(z) = f(z) + \bar{f}(z) \in \mathbb{R}$, i.e. its imaginary part $\psi = 0$ on the boundary.

So taking $U \in \mathbb{R}$ and $f(z) = -Uz$, then $\frac{d\bar{z}}{dt} = -\frac{df}{dz} = U + 0.i$ which is the velocity field we want. We have to add the contribution of the virtual field $f^*(1/z) = -U\frac{1}{z}$ outside the unit circle.

Finally the total complex potential describing this model is

$$w(z) = -U\left(z + \frac{1}{z}\right) + i\frac{\gamma_1}{2\pi}\left[\log(z - z_1) - \log(z - z_2) - \log(z\bar{z}_1 - 1) + \log(z\bar{z}_2 - 1)\right] \quad (2.50)$$

and thus if $\frac{dz_1}{dt} = u_1 + iv_1$,

$$-u_1 + iv_1 = \frac{d\hat{w}^1}{dz}(z_1) = -U\left(1 - \frac{1}{z_1^2}\right) + i\frac{\gamma}{2\pi}\left[-\frac{1}{z_1 - z_2} - \frac{\bar{z}_1}{z_1\bar{z}_1 - 1} + \frac{\bar{z}_2}{z_1\bar{z}_2 - 1}\right] \quad (2.51)$$

and similarly for $\dot{z}_2$. It is realised that if initially $z_1(0) = \bar{z}_2(0)$, since $u_1 = u_2$ and $v_1 = -v_2$ $\forall\ z_1 = \bar{z}_2$, the symmetry is conserved at all times. This leads us to limit the study of the system to a completely symmetric case, we derive the equation of motion of z_1 on the upper half-plane knowing

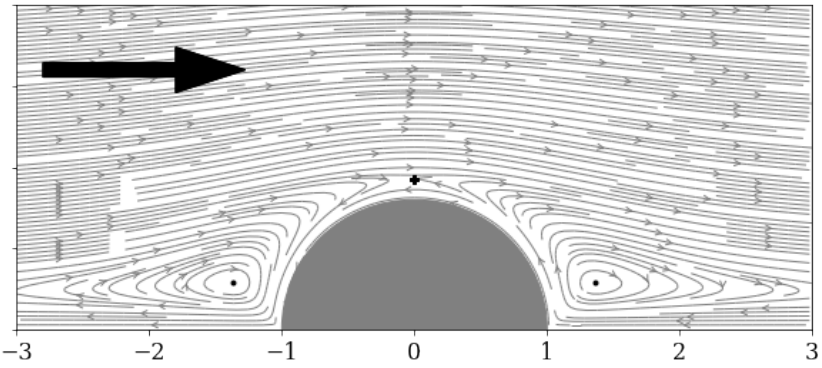


Figure 2.4: Phase portrait of the system for the upper half-plane. We can see the center Föppl equilibria upstream and downstream (black dots), and the unstable fixed point on the vertical line (bold cross). It shows the possible motion of a single vortex put in the uniform flow.

that the motion of z_2 will always be symmetric. We are interested in the stable configurations of point vortices on both sides of the cylinder, and therefore we will set $\gamma < 0$ (otherwise there would not be any in the upper plane). We write $z = z_1 = \bar{z}_2$, $u + iv = u_1 + iv_1$, re-scaling the time variable

$$\begin{aligned} t \rightsquigarrow U.t \quad &\Rightarrow w = -\bar{\dot{z}} \rightsquigarrow -\bar{\dot{z}}/U = w/U, \quad \text{and defining } \kappa := -\frac{\gamma}{2\pi U} > 0 \\ \Rightarrow -u + iv &= -\left(1 - \frac{1}{z^2}\right) + i\kappa\left[\frac{1}{z-\bar{z}} + \frac{\bar{z}}{z\bar{z}-1} - \frac{z}{z^2-1}\right] \end{aligned} \tag{2.52}$$

In this equation the dimensionless parameter κ is the intensity of the vortices. In terms of x, y, if $z = x + iy$ ($z_2 = x - iy$), defining $r^2 := z\bar{z} = x^2 + y^2 \in \mathbb{R}$,

$$-u + iv = -\left(1 - \frac{x^2 - y^2 - 2ixy}{r^4}\right) + i\kappa\left[\frac{1}{2iy} + \frac{x - iy}{r^2 - 1} - \frac{x(r^2-1) - iy(r^2+1)}{(r^2-1)^2 + 4y^2}\right] \tag{2.53}$$

$$\Rightarrow \begin{cases} u = 1 - \dfrac{x^2 - y^2}{r^4} - \dfrac{\kappa}{2}.\dfrac{1}{y} + \kappa y\left(-\dfrac{1}{r^2-1} + \dfrac{r^2+1}{(r^2-1)^2 + 4y^2}\right) \\ v = \dfrac{-2xy}{r^4} + \kappa x\left(\dfrac{1}{r^2-1} - \dfrac{r^2-1}{(r^2-1)^2 + 4y^2}\right) \end{cases} \tag{2.54}$$

This coupled equation for $u = \dot{x}, v = \dot{y}$ gives the phase portrait Figure 2.4.

Let us study the fixed points of this system. A point is fixed when $\dot{x} = 0, \dot{y} = 0$, its location depends on κ too.

– First, we see that letting $x \to \pm\infty$, we have an equilibrium at $y = \kappa/2$.

– Next, since on the vertical line $x = 0$ we necessarily have $v = 0$, so $u = 0$

$$\begin{aligned} &\Leftrightarrow\ 1 + \frac{1}{y^2} - \frac{\kappa}{2}.\frac{1}{y} + \kappa y\left(-\frac{1}{y^2-1} + \frac{y^2+1}{(y^2+1)^2}\right) = 0 \\ &\Rightarrow\ \kappa = \frac{2(y^2+1)^2(y^2-1)}{y(y^4+4y^2-1)} \end{aligned}$$

For each vertical location $y > 0$ chosen, there is an intensity for which $(0, y)$ is a fixed point. It is seen in the phase portrait at the intersection of the opposing field lines on the vertical axis.

– Finally the locus of equilibria that interest us, (x_0, y_0) such that

$$r_0^2 - 1 = 2x_0y_0, \qquad \text{of intensity } \kappa = \frac{(r_0^2+1)(r_0^2-1)^2}{r_0^5}, \text{ where } r_0^2 := x_0^2 + y_0^2. \tag{2.55}$$

These are called the *Föppl equilibria*, found on the *Föppl curve* $r_0^2 - 1 = 2x_0y_0$. The further the couple of symmetric point vortices is placed ($r_0 \to \infty$), the greater their opposing intensity must be ($\kappa \to \infty$).

The stability of those equilibrium points can then be analysed. Vasconcelos et al. (2011) finds that for an infinitesimal symmetric perturbation around the equilibrium, the linearised system has two purely imaginary eigenvalues, and thus the Föppl points are locally centers. In other words, when a vortex point is slightly displaced from the fixed point, it will execute periodic curves around the point (an so will its symmetric on the lower half-plane).

To be noted also that with the definition (2.55), at each set intensity there are two possible fixed points: One with $x_0 < 0$, said to be upstream, and the other with $x_0 > 0$, downstream. The upstream equilibrium for point vortices is observed in nature when stationary eddies form before an obstacle when fluid flows past it.

The paper goes further and determines the properties of the Föppl Equilibrium also in the case of a non-symmetric dynamic (using then expression 2.51). In the symmetric case, these points are relatively stable, in the sense that a slightly perturbed trajectory stays at finite distance of the equilibrium point. It is no longer the case if there is a non-symmetric perturbation, the Föppl fixed point becomes unstable and even small perturbations from it will cause the points to get away. It is thought that this instability gives rise to the phenomenon of Karman Street (Vasconcelos et al., 2011), that is observed downstream a circular obstacle in nature. A couple of lines of evenly spaced point vortices is created, moving uniformly along with the fluid. However our model does not explain the process of creation of new vortices, since one of our hypothesis was that the vorticity is transported. The creation of those vorticity points coming from the presence of the cylinder is only possible in the presence of the viscosity term. The fluid then does not follow the Euler equation, in that case the vorticity is no longer conserved along streamlines and indeed new vortices are produced, thanks to the dissipation due to viscosity. △

Example 2.6 (Multiply connected domains). Now that we have seen a concrete example of multiply connected domain, we are interested in the general case of bounded domains with holes. In the same way that the circle is fundamental to study a lot of other simply connected domains, considering a circle with M circular holes in it is the typical region, that can then be generalised on other multiply connected domains of connectivity M.

That is why we will study the motion of a single vortex inside a domain consisting in a big disc (say the unit circle), written $\mathcal{C}_0$, containing M smaller circular boundaries $\mathcal{C}_1, \dots \mathcal{C}_M$. We are interested in the expression of the Hamiltonian in this case. The general derivation and more discussions are presented in Crowdy and Marshall (2005), here we will go through the article and give the ideas behind the formulas without much details.

First of all, since we will often need to refer to the symmetrical of a point through a circle, we define the functions

$$\mathcal{C}_0(z) := z^* = 1/\bar{z} \qquad \text{Symmetric through the unit circle } \mathcal{C}_0\text{, and in general}$$
$$\mathcal{C}_j(z) := c_j + \frac{R_j^2}{\bar{z} - \bar{c}_j} \qquad \text{through the circle } \mathcal{C}_j \text{ of center } c_j \text{ and radius } R_j.$$

It can be checked that a circle inside (resp. outside) C_j of center α and radius δ is sent through a function $\mathcal{C}_j$ to another circle, outside (resp. inside) $\mathcal{C}_j$, of center $c_j + \frac{R_j^2}{|z-c_j|^2-\delta^2}(z - c_j)$ and radius $\frac{R_j^2\delta}{|z-c_j|^2-\delta^2}$.

For a point vortex, the interaction with its symmetrical has to be added when the solid boundary $w = 0$ is imposed. In the same way here we have to consider the action of that symmetric $\mathcal{C}_0(z)$, but also the action of the symmetric points through the empty domains $\mathcal{C}_j$, of the images of those circles through each other and the greater one, etc. The idea is to add the interactions with an infinity of points, and circles, distributed in the plane such that the velocity field does not cross any boundary. Those points will be the reflection of the initial point vortex z_1 in the circles, then the reflection of those reflections, etc. The whole contribution is obtained by action of what is called the *Schotty group* on the initial vortex z_1.

To get to the expression of the Hamiltonian, we start by defining the *Möbius maps* associated to the circle C_j,

$$\theta_j(z) := c_j \frac{R_j^2 z}{1 - \bar{c}_j z} \tag{2.56}$$

such that $\theta_j(\bar{z}^{-1}) = \mathcal{C}_j(z)$. The Schotty group Θ is then defined as the transformation group spanned by the $\{\theta_j : j = 1, \dots M\}$. We also write Θ'' the subset of the group consisting of all the composition of mappings without the identity and all the inverses,

$$\Theta'' = \{\theta_1, \dots \theta_M, \theta_1 \circ \theta_1, \theta_1 \circ \theta_2, \dots \theta_M \circ \theta_M, \dots \theta_1 \circ \cdots \theta_2, \dots\}. \tag{2.57}$$

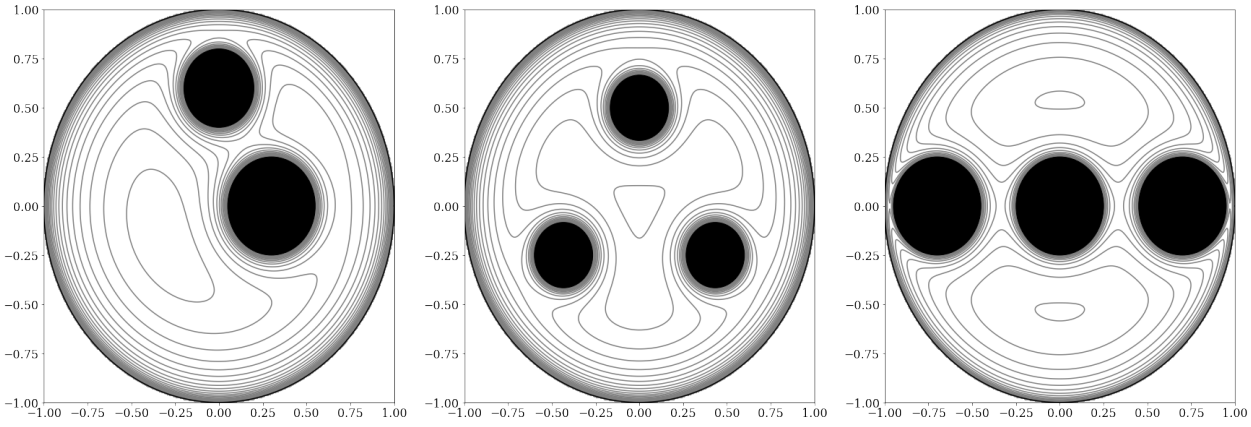


Figure 2.5: The stream curves in various multiply connected domains.

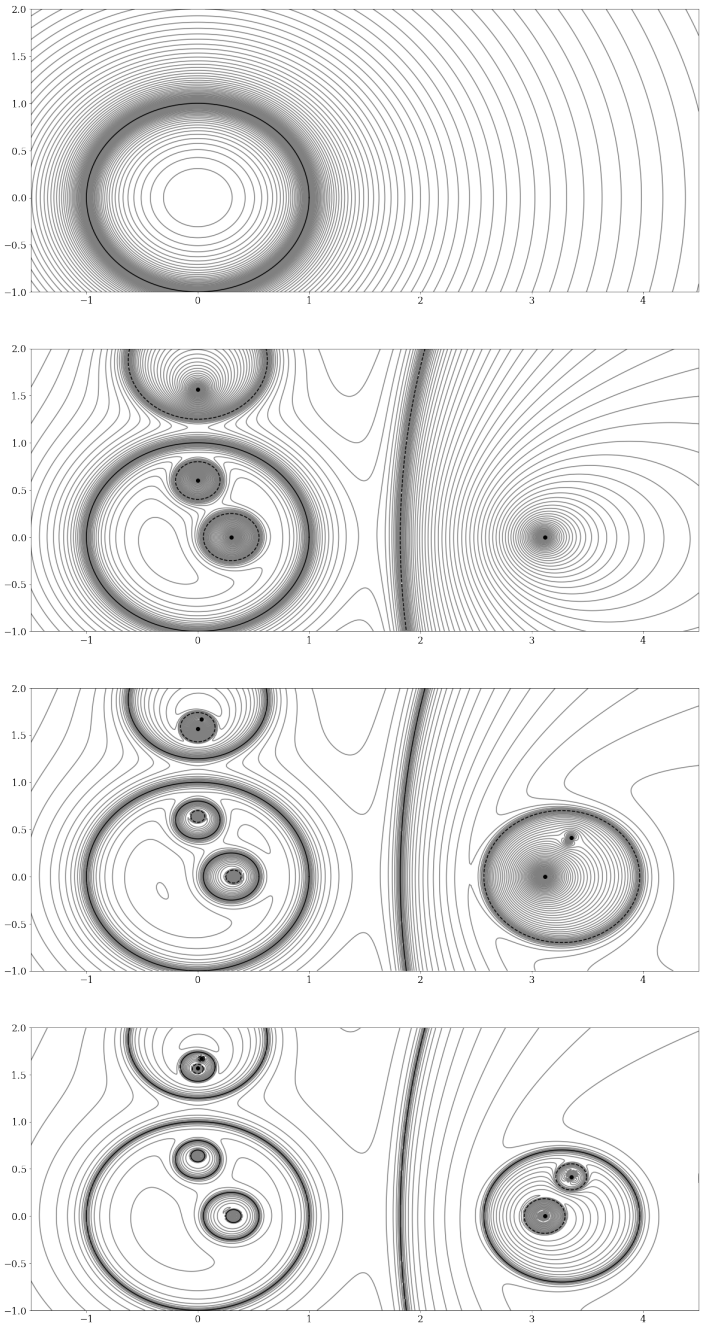


Figure 2.6: Level curves of the Hamiltonian for $\theta_i \in \Theta$ until 0,1,2 and 4 compositions of the basic Möbius maps. (a) $\Theta_0 = \{id\}$ and the Hamiltonian is simply the one of a single point in the circle. (b) Depth 1, $\Theta_1'' = \{\theta_1, \theta_2\}$, we see the apparition of the circles $\mathcal{C}_j$ and their image $\mathcal{C}_j'$. (c) Depth of 2, singularities appear at the image of each center, $\mathcal{C}_j'(c_1), \mathcal{C}_j'(c_2)$ and the apparition of the big circle $\mathcal{C}_0$ through the circles and their image. (d) Depth of 4, the dashed lines correspond to the image through $\mathcal{C}_j'$ of the little circles $\mathcal{C}_j$.

. Then we have

$$H(z_1) = -\frac{1}{2}G_D^{(1)} = \frac{\gamma}{4\pi}\log|1 - z_1\bar{z}_1| + \frac{\gamma}{8\pi}\log\left|\omega(z_1, \bar{z}_1^{-1})\omega^*(z_1^{-1}, \bar{z}_1)\right| \tag{2.58}$$

$$\text{where } \omega(\zeta,\eta) := \prod_{\theta_i\in\Theta''} \frac{(\theta_i(\zeta)-\eta)(\theta_i(\eta)-\zeta)}{(\theta_i(\zeta)-\zeta)(\theta_i(\eta)-\eta)} \tag{2.59}$$

$$\text{and } \omega^*(\zeta,\eta) := \bar{\omega}(\bar{\zeta},\bar{\eta}). \tag{2.60}$$

The first term in the Hamiltonian is the contribution of the big circle surrounding all the others, whereas the other terms add the contribution of all the images going through Θ''. If we write Θ_K the finite subset of all compositions of at most K basic Möbius maps, such that for example $\Theta_2'' = \{\theta_1, \dots \theta_M, \theta_1 \circ \theta 1, \dots \theta_1 \circ \theta_M, \dots \theta_M \circ \theta_M\}$, then each level of Θ_K add more and more details to the global Hamiltonian (Figure 2.6).

If no internal circle is added, $M = 0$ and we expect H to have the expression of the Example 2.4. Indeed in that case $\Theta = \{id\}$ and so $\omega = 1$, and the formula works. Applying Equation (2.58) to various examples, and assuming that the product (2.59) converges, we get the flow lines that are the level curves of H (Figure 2.5). △

2.3 Curved Surfaces

Now that we have studied the effects of boundaries on the motion of point vortices, we are interested in the generalisation of our equations on curved surfaces. In other words we want to see how the equation transforms if we consider that it takes place onto a two-dimensional manifold. First, we will see how to rewrite the equations in differential coordinate-free form. In a second step, we will try to solve it and get Hamiltonian forms describing the motion of point vortices on the Sphere and hyperbolic plane, and compare its properties to the euclidean plane we already studied. This part of the section comes from Kimura (1999). Finally we will see some notable periodic solutions on the Sphere, described by Tokieda (2001).

2.3.1 Deriving the equations

The interest in curved surfaces comes from physics. For example, the sphere $\mathcal{S}^2$ can serve as a basis for the study of vortices on earth, in the atmosphere, or on surfaces of water sufficiently extended and supposed at rest. Moreover, it is interesting to compare the properties of motion on surfaces with different curvatures. The sphere being the canonical example of constant positive Gaussian curvature, we are also interested in the hyperbolic plane $\mathcal{H}^2$ which has constant negative curvature.

Let us start by reminding the equations we had on the Euclidean plane:

$$\omega := \frac{\partial u^2}{\partial x} - \frac{\partial u^1}{\partial y} \qquad \mathbf{u} = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right) \qquad \Rightarrow \omega = \Delta\psi \,(= \operatorname{div} \mathbf{grad}\ \psi) \tag{3.1}$$

On $\mathbb{R}^2$, we then used the fundamental solution $G(z) = \frac{1}{2\pi}\log|z|$ of the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Thus we had

$$\begin{aligned} \mathbf{u}(z,t) &= \nabla^\perp\psi = \nabla^\perp G \star \omega = \sum_j \gamma_j \nabla^\perp G(z; z_j) \\ \dot{z}_i(t) &= \hat{\mathbf{u}}^i(z_i,t) = \sum_{j\neq i} \gamma_j \nabla^\perp G(z_i; z_j) \end{aligned} \tag{3.2}$$

We could use the same arguments and get $\dot{z}_i$ if we know the expression of the solution ψ of Equation (3.1) on a manifold $\mathcal{M}$. Unfortunately, this equation does not have a tensorial form, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is not an invariant expression by change of coordinates on any surface. We must then define a generalisation of the Laplacian that is independent of the coordinate system we choose to describe our manifold, it is given by the *Laplace-Beltrami operator*

$$\Delta := \sum_{i,j} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left[\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right] \qquad g := \det(g_{ij}) \tag{3.3}$$

where (g_{ij}) is the Riemannian metric of the manifold, and g^{ij} its inverse. This expression is obtained realising that **grad** and div can be expressed in a coordinate-free way on a m-dimensional manifold via

$$\textbf{grad} \ : \ \underbrace{0\text{-forms}}_{=\text{ scalar fields}} \to \underbrace{1\text{-forms}}_{\simeq \text{ vectors}} : f \mapsto df \tag{3.4}$$

$$\text{div} : \underbrace{(m-1)\text{-forms}}_{\simeq \text{ vectors}} \to \underbrace{m\text{-forms}}_{\simeq \text{ scalar fields}} : \Upsilon \mapsto d\Upsilon. \tag{3.5}$$

And so by the fundamental duality between k-forms and $(m-k)$-forms ($\chi \mapsto {}^*\chi$) we can define

$$\Delta : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}) : f \mapsto {}^*\text{div } {}^*\textbf{grad } f \tag{3.6}$$

which has the form (3.3) when expressed in some coordinates (x^i) with metric g_{ij}.

We want to solve the problem $\Delta G = \delta$, and then be able to express $\Delta\psi = \omega$ as

$$\psi(z) = \int_{\mathcal{M}} G(z; z')\omega(z')dS(z') \tag{3.7}$$

It is not an easy problem on a general manifold.

On the Sphere : We are placed at the surface of a sphere, say of radius 1, it forms a manifold of dimension 2. The position of a point can then be described with the two angles $\theta \in [0, \pi]$ the polar angle, or inclination, and $\varphi \in [0, 2\pi]$ the azimuthal angle, or longitude. In those coordinates the metric on the surface is $ds^2 = d\theta^2 + \sin^2\theta d\varphi^2 = \sum_{i,j} g_{ij} d\zeta^i d\zeta^j$, writing $\zeta = (\theta, \varphi)$. The Laplace-Beltrami operator has the expression

$$\Delta = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \tag{3.8}$$

One can directly check that for $\theta \in (0, \pi)$, $u(\theta, \varphi) = c \log(\tan(\frac{\theta}{2}))$ is harmonic, $\Delta u = 0$ for c constant. But there is a problem, this function has two singularities, one at each pole $\theta = 0$ and $\theta = \pi$. We want the Green function to have only one singularity, that we will place on the location of the vortex when convoluting with ω. Let us then remove the division by $\cos\theta/2$ that causes this, and define

$$\mathbf{u}(\theta, \varphi) = c \log(\sin(\theta/2)), \quad \theta \in (0, \pi]. \tag{3.9}$$

There is no longer a singularity at the other pole $\theta = \pi$, but now $\Delta u = -\frac{c}{2}$. We also know that we must take $c = \frac{1}{2\pi}$ to get the delta distribution at $\theta = 0$.

This comes to solving $\Delta G = \delta - \frac{1}{4\pi}$ on $\mathcal{S}^2$, we had to add a background vorticity potential, but it will not influence the motion of the vortices since it is everywhere the same. The periodicity of the coordinate θ causes the Δ operator to no longer be invertible, and so we need to modify the operator to find a suitable Green function, which is in some sense the inverse operator.

From this solution, if we place the singularity at $\zeta_1 = (\theta_1, \varphi_1)$ and we denote by ρ the angle between (θ, φ) and (θ_1, φ_1) (equal to the length of the arc joining ζ and ζ_1 on the sphere), by rotational symmetry we find the kernel

$$G(\zeta; \zeta_1) = \frac{1}{2\pi} \log \sin \rho/2. \tag{3.10}$$

Green function on the sphere (see Remark below). We have the expression for ρ,

$$\cos(\rho(\theta_1, \varphi_1, \theta_2, \varphi_2)) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2). \tag{3.11}$$

So using that $\sin^2 \alpha = \frac{1-\cos 2\alpha}{2}$ we take

$$G(\zeta; \zeta_1) = \frac{1}{4\pi} \log\left(\frac{1-\cos\rho}{2}\right). \tag{3.12}$$

As always we consider N point vortices $\zeta_1, \dots \zeta_N$ of intensities $\gamma_1, \dots \gamma_N$, then we have

$$\begin{aligned} \psi(\theta, \varphi) &= \frac{1}{4\pi} \int_{\mathcal{S}^2} \omega(\theta', \varphi') \log\left(\frac{1-\cos\rho(\zeta, \zeta')}{2}\right) \sin(\theta') d\theta' d\varphi' \\ &= \frac{1}{4\pi} \sum_i \gamma_i \log\left(\frac{1-\cos\rho(\theta, \varphi, \theta_i, \varphi_i)}{2}\right) \sin(\theta_i) \end{aligned} \tag{3.13}$$

Remark : Let us see if it works, we must have $\Delta(G \star \omega) = \omega$, but

$$\begin{aligned} \Delta(G \star \omega) &= \Delta G \star \omega = (\delta - \frac{1}{4\pi}) \star \omega \\ &= \omega - \frac{1}{2\pi} \int_{\mathcal{S}^2} \omega dS \end{aligned} \tag{3.14}$$

So the choice of G allowing to avoid creating another vortex at the antipodal position forces a condition on ω:

$$\begin{aligned} &\int_{\mathcal{S}^2} \omega \; dS = 0 \\ &\qquad = \int_{\theta\in[0,\pi]} \int_{\varphi\in[0,2\pi]} \sum_i \gamma_i \delta_{\zeta_i}(\theta, \varphi) d\theta \sin\theta d\varphi = \sum_i \gamma_i \sin(\theta_i) \end{aligned} \tag{3.15}$$

And since $\sin\theta$ is always positive, we cannot set up a system of point vortices all with vorticity of the same sign. In a sense the contribution of vorticity in the upper hemisphere must be balanced by the one in the lower half. This comes from the fact that the sphere is a bounded domains without boundaries, and so the total vorticity must satisfy this condition to make sense in the whole domain.

Now we have $\mathbf{u} = \nabla^\perp \psi = J.\mathbf{grad}\ \psi$, J is a matrix rotating the vector by a quarter turn clockwise. Expressed in an orthonormal basis of the tangent vector space, $(\mathbf{e}_j)$,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.16}$$

We must express our vectors in that basis of unit vectors, $\partial_j \rightsquigarrow \mathbf{e}_j := \frac{1}{\sqrt{g_{jj}}}\partial_j$,

$$\mathbf{grad}\ \psi = \sum_{i,j} g^{ij}\frac{\partial\psi}{\partial\zeta^j}\partial_i = \sum_{i,j} g^{ij}\frac{\partial\psi}{\partial\zeta^j}\sqrt{g_{ii}}\ \mathbf{e}_i = \frac{\partial\psi}{\partial\theta}\ \mathbf{e}_\theta + \frac{1}{\sin^2\theta}\sin\theta\frac{\partial\psi}{\partial\varphi}\ \mathbf{e}_\varphi \tag{3.17}$$

$$\mathbf{u} = u_\theta\ \mathbf{e}_\theta + u_\varphi \sin\theta\ \mathbf{e}_\varphi. \tag{3.18}$$

Removing once again the self-interacting term, we obtain the motion equations

$$\begin{aligned}\frac{d\theta_i}{dt} &= -\frac{1}{4\pi}\sum_{j\neq i}\gamma_j\frac{1}{\sin\theta_i}\frac{\sin\theta_i\sin\theta_j\sin(\varphi_i-\varphi_j)}{1-\cos\rho_{ij}} = -\frac{1}{4\pi}\sum_{j\neq i}\gamma_j\frac{\sin\theta_j\sin(\varphi_i-\varphi_j)}{1-\cos\rho_{ij}}\\ \sin\theta_i\frac{d\varphi_i}{dt} &= \frac{1}{4\pi}\sum_{j\neq i}\gamma_j\frac{\sin\theta_i\cos\theta_j-\cos\theta_i\sin\theta_j\cos(\varphi_i-\varphi_j)}{1-\cos\rho_{ij}}.\end{aligned} \tag{3.19}$$

On the Hyperbolic plane We have derived the motion equation for point vortices on the sphere, which is a curved surface of constant Gaussian curvature 1. Equivalently on $\mathcal{H}^2$, parameterized like $\mathcal{S}^2$ but with a purely imaginary polar angle. All $\sin\theta, \cos\theta$ become hyperbolic, $\sinh\theta, \cosh\theta$, $\theta \in [0,\infty)$. The metric is $ds^2 = d\theta^2 + \sinh^2\theta d\varphi^2$ and the Laplace-Beltrami operator becomes

$$\Delta = \frac{1}{\sinh\theta}\frac{\partial}{\partial\theta}\left(\sinh\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sinh^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{3.20}$$

A solution of the Laplace equation $\Delta u = 0$ is $u(\theta,\varphi) = c\log(\tanh(\frac{\theta}{2}))$ on $\theta \in (0,\infty)$ and has a singularity at $\theta = 0$. By symmetry once again if we place the singularity at (θ_1,φ_1),

$$G(\zeta;\zeta_1) = \frac{1}{2\pi}\log(\tanh(\rho/2)) \tag{3.21}$$

with ρ the "hyperbolic arc" length between (θ,φ) and (θ_1,φ_1), i.e. the hyperbolic distance from the geodesics on $\mathcal{H}^2$,

$$\cosh(\rho(\theta_1,\varphi_1,\theta_2,\varphi_2)) = \cosh\theta_1\cosh\theta_2 - \sinh\theta_1\sinh\theta_2\cos(\varphi_1-\varphi_2). \tag{3.22}$$

We use the identity $\tanh^2\alpha = \frac{\cosh 2\alpha - 1}{\cosh 2\alpha + 1}$. Then

$$G = \frac{1}{4\pi}\log\left(\frac{\cosh\rho - 1}{\cosh\rho + 1}\right) \tag{3.23}$$

$$\text{and } \psi = \frac{1}{4\pi}\sum_j \gamma_j \log\left(\frac{\cosh\rho(\theta,\varphi,\theta_i,\varphi_i) - 1}{\cosh\rho(\theta,\varphi,\theta_i,\varphi_i) + 1}\right)\sinh(\theta_i). \tag{3.24}$$

The motion equations are

$$\begin{aligned}\frac{d\theta_i}{dt} &= -\frac{1}{\sinh\theta_i}\frac{\partial\psi}{\partial\varphi} = -\frac{2}{4\pi}\sum_{j\neq i}\frac{\sinh\theta_j\sin(\varphi_i-\varphi_j)}{\cosh^2\rho_{ij}-1}\\ \sinh\theta_i\frac{d\varphi_i}{dt} &= \frac{\partial\psi}{\partial\theta} = \frac{1}{2\pi}\sum_{j\neq i}\gamma_j\frac{\sinh\theta_i\cosh\theta_j - \cosh\theta_i\sinh\theta_j\cos(\varphi_i-\varphi_j)}{\cosh^2\rho_{ij}-1}.\end{aligned} \tag{3.25}$$

Remark : On an arbitrary surface, the reasoning still holds, but an explicit form of G, ψ and so the motion equation may be very difficult to derive.

2.3.2 Some solutions

We realise that both of those equations (3.19 and 3.25) describe a Hamiltonian mechanic, defining

$$H = -\frac{1}{4\pi}\sum_{i\neq j}\gamma_i\gamma_j \log(1-\cos\rho_{ij}) \qquad \text{on } \mathcal{S}^2, \tag{3.26}$$

$$H = -\frac{1}{4\pi}\sum_{i\neq j}\gamma_i\gamma_j \log\left(\frac{\cosh\rho_{ij}-1}{\cosh\rho_{ij}+1}\right) \qquad \text{on } \mathcal{H}^2. \tag{3.27}$$

And using the structure of Poisson bracket defined as

$$\{f,g\} := \sum_{k=1}^{N}\frac{1}{\gamma_k}\left[\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k} - \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k}\right] \tag{3.28}$$

$$\begin{aligned} \text{where } p_k = \varphi_k, \quad q_k = \cos\theta_k, \qquad & \text{on } \mathcal{S}^2, \\ p_k = \varphi_k, \quad q_k = -\cosh\theta_k, \quad & \text{on } \mathcal{H}^2, \end{aligned} \tag{3.29}$$

we can verify that $\dot\theta_i = \{\theta_i, H\}$ and $\dot\varphi_i = \{\varphi_i, H\}$. Using this we can then find first integrals F_k, that must satisfy $\{F_k, H\} = 0$, i.e.

$$c_1 = \sum_j \gamma_j \sin\theta_j\cos\varphi_j \qquad c_2 = \sum_j \gamma_j \sin\theta_j\sin\varphi_j \qquad c_3 = \sum_j \gamma_j\cos\theta_j \qquad \text{on } \mathcal{S}^2, \tag{3.30}$$

$$d_1 = \sum_j \gamma_j \sinh\theta_j\cos\varphi_j \qquad d_2 = \sum_j \gamma_j \sinh\theta_j\sin\varphi_j \qquad d_3 = \sum_j \gamma_j\cosh\theta_j \qquad \text{on } \mathcal{H}^2. \tag{3.31}$$

where (c_1, c_2, c_3) and (d_1, d_2, d_3) are respectively the coordinates of $M = \Gamma m$, m being the center of vorticity expressed in the euclidean space $\mathbb{R}^3$ in which $\mathcal{S}^2$ and $\mathcal{H}^2$ are embedded, so these constants of motion tell that the center of vorticity is conserved during the vortex motion on the Sphere and Hyperbolic plane. We also note that the constant of movement $\sum_j \gamma_j |z_i|^2$ has no equivalent in these surfaces, in contrast with the Euclidean plane.

Finally we can look, for the sphere's case, at several examples.

Example 3.1 (Two points). Let us take two point vortices (θ_1, φ_1), (θ_2, φ_2) of vorticities $\gamma_1 = -\gamma_2 := 4\pi\gamma$. The conserved quantities can be written

$$A = \sin\theta_2\cos\varphi_2 - \sin\theta_1\cos\varphi_1, \quad B = \sin\theta_2\sin\varphi_2 - \sin\theta_1\sin\varphi_1, \quad G = \cos\theta_2 - \cos\theta_1. \tag{3.32}$$

Starting with a couple of particles symmetric with respect to the equator, $\theta_2 = \pi - \theta_1, \varphi_2 = \varphi_1$ (then we do have $\sum\gamma_i\sin\theta_i = 0$). We have $A = B = 0$ and the motion equation yields

$$\begin{aligned} \theta_1(t) &= \theta_1(0) \\ \varphi_1(t) &= \frac{\gamma}{\cos\theta_1(0)}t + \varphi_1(0). \end{aligned} \tag{3.33}$$

The vortex pair turns following the equator, at constant angular speed.

By symmetry of rotation it is true for all pair of points starting symmetrically on each side of a geodesic. It is the spherical analog of the Equation (1.4), but here the motion becomes periodic instead of unbounded. △

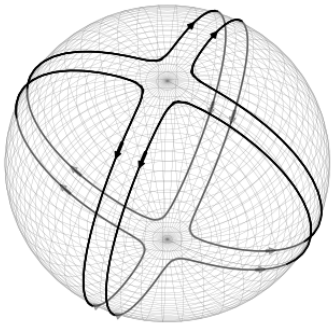


Figure 3.7: The motion of four point vortices in a particular configuration on the sphere.

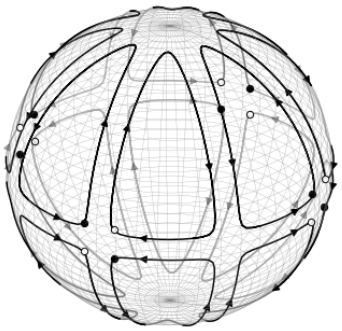


Figure 3.8: The motion of sixteen vortices, starting with four groups of vortices.

Example 3.2 (Periodic Solutions). We start by constructing a periodic motion with four vortices. Taking $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ with vorticities $0 < \gamma_1 = -\gamma_2 = \gamma_3 = -\gamma_4$, and starting on the equator separated in two couples (z_1, z_2 and z_3, z_4) of very close vortices with opposed vorticities, the two couples being diametrically opposed. In this configuration each couple only feels a weak influence coming from the other, negligible when considering the effect of the near vortex. Each couple then moves straight up towards the North pole, and doing so gets closer to the other one. When the effects of the other vortices become relevant the vortices start to curve their motion and finally exchange partner. They then descend towards the equator and then by symmetry of the system they will follow the same path than in the beginning, but reversed. They get down to the South pole and exchange once again to regain their initial partner, and so on (Figure 3.7).

This motion derives entirely from the symmetry of the sphere and the initial configuration. It is also notable that the equations of movement have a symmetry $t \rightsquigarrow -t, \gamma_i \rightsquigarrow -\gamma_i \ \forall \ i$, so that the dynamic of the four vortices is the same rotating the initial sphere and then reversing the intensities.

We can then generalise this construction, for $2N$ vortices of alternating intensities. Starting from a regular N-gon of couple of vortices with opposed vorticities on the equator, i.e. a pair of N-gons slightly rotated (in the previous example we had a pair of 2-gons, a line). The reasoning above holds and they will follow a similar path than the two pair, only cutting the sphere in more slices. By symmetry once again, we take this time $4N$ vortices, reproducing the $2N$ vortices but doubled, one slightly over the equator and one slightly below (see Figure 3.8).

On the other hand, the Hyperbolic plane does not have such closed periodic trajectories, and the only simple ones are thus the relative equilibria of N vortices on the vertices of a regular shape centered at the origin, all with same positive vorticity. The system will then rotate on itself without changing shape nor size. △

References

Crowdy, D., & Marshall, J. (2005). Analytical formulae for the Kirchhoff-Routh path function in multiply connected domains. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, *461*(2060), 2477–2501. doi: 10.1098/rspa.2005.1492

Kimura, Y. (1999). Vortex motion on surfaces with constant curvature. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, *455*(1981), 245–259. doi: 10.1098/rspa.1999.0311

Majda, A. J., & Bertozzi, A. L. (2002). *Vorticity and incompressible flow* (Vol. 27). Cambridge University Press, Cambridge.

Marchioro, C., & Pulvirenti, M. (1994). *Mathematical theory of incompressible nonviscous fluids* (Vol. 96). Springer-Verlag, New York. doi: 10.1007/978-1-4612-4284-0

Miot, E. (2015). Le système dynamique de N tourbillons ponctuels. In *Des problèmes à N corps aux tokamaks* (pp. 27–53). Ed. Éc. Polytech., Palaiseau.

Newton, P. K. (2001). *The N-vortex problem* (Vol. 145). Springer-Verlag, New York. (Analytical techniques) doi: 10.1007/978-1-4684-9290-3

Tokieda, T. (2001). Tourbillons dansants. *C. R. Acad. Sci. Paris Sér. I Math.*, *333*(10), 943–946. doi: 10.1016/S0764-4442(01)02162-0

Vasconcelos, G. L., Moura, M. N., & Schakel, A. M. J. (2011). Vortex motion around a circular cylinder. *Physics of Fluids*, *23*(12).