Mémoire

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Asymptotic analysis of critical conformal equations of fourth order

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Preface

The interest in fourth-order, bi-Laplacian equations arises naturally in the study of partial differential equations, on one hand from the physical standpoint, and on the other hand from a purely mathematical point of view as a way to generalize known results in second-order theory. This Master's Thesis focuses on the study of elliptical equations of fourth order on smooth manifolds. In particular, motivated by concentration results for critical Laplacian equations in the Yamabe problem as developed in Struwe (2008), we investigate the creation of bubbles preventing the sequences of solutions to bi-Laplacian equations from converging strongly. These defaults of compactness are studied as they present a particular well-known shape, the purpose is to obtain sharp estimates on the asymptotic behavior of sequences of solutions that blow up in L^∞-norm, and understand under which conditions this explosion can occur. We also spend some time discussing the discrepancies there are when going from the second order to the fourth. The main reference of the work has been the article of Hebey and Robert (2011), completed by filling in the items that were skipped in some proofs and adding insight and further results coming from complementary literature.

The foundations of this work rely heavily on the theory of Sobolev spaces on $\mathbb{R}^n$. Thus a basic knowledge of this branch and some familiarity with existence and regularity theory in the context of second-order equations may help in understanding the arguments. More generally, a foundation of functional analysis and variational methods is useful to be able to grasp all the concepts that will be discussed. Although all the necessary notions will be recalled in the first two chapters in the context of smooth manifolds, most of the proofs will be omitted to dive into the main problem of the work. In this second part, consisting of Chapters III and IV, the goal is to gather and compile the research work of Y. Wen, F. Robert and E. Hebey in particular, to learn the state-of-the-art results in the asymptotic analysis of solutions to a critical non-linear equation of Paneitz-type, and finally reach some stability results. This Master's Thesis was written as part of the Master in Mathematics, Research Focus, of the Université Libre de Bruxelles.

Throughout this work, we consider a smooth compact Riemannian manifold (M, g) of dimension n without boundary. We are interested in equations of the kind

$$\Delta_g^2 u + b\Delta_g u + cu = fu^{2^\# - 1} \qquad \text{in } M, \tag{1}$$

where $f \in C^\infty(M)$, Δ_g denotes the Laplace-Beltrami operator on M as defined in Section I.1, $b^2 - 4c > 0$ so that the maximum principle applies, and $2^\# = \frac{2n}{n-4}$ the critical exponent for the embedding of $H^2(M)$ into $L^p(M)$. This type of equation naturally emerges when considering a prescribed Q-curvature on a Riemannian manifold through conformal changes of the metric. We will see in Section II.1 some particular expressions of this equation in several cases, and specifically we obtain the shape of (1) in the case of an Einstein metric. Moreover, in Chapter I we justify the critical appellation, and we introduce the concept of weak solutions $u \in H^2(M)$ to partial differential equations of this kind.

In the second chapter, we obtain existence, unicity and positivity results for solutions to (1). The interest of this Master's Thesis relies more specifically in discussing the stability of such equations, that is to have necessary or sufficient conditions on the value of the coefficients in (1) for the solution set to be compact in $C^4(M)$-topology. To understand what we mean here by stability we perturb the Equation by taking sequences (b_α), (c_α) of coefficients converging respectively to b and c, and consider a bounded sequence $(u_\alpha)_\alpha$ of solutions in $H^2(M)$ to

$$\Delta_g^2 u + b_\alpha \Delta_g u + c_\alpha u = u^{2^\# - 1}. \tag{2}$$

The question is to know whether $(u_\alpha)_\alpha$ admits a limit $u_\infty \in C^4(M)$ that solves (1) such that $u_\alpha \to u_\infty$ in $C^4(M)$.

To answer this, we will need to understand the apparition of bubbles that can occur when dealing with critical equations of this shape. Although in the subcritical case, when looking at equations of the shape

$$\Delta_g^2 u + b_\alpha \Delta_g u + c_\alpha u = u^{p-1} \tag{3}$$

with $p < 2^\#$, regular solutions can be obtained through the minimisation of an energy functional and the equation is stable, in the critical case bounded sequences of solutions can loose compactness. As long as we work in the weak space $H^2(M)$, we know exactly how the sequence behaves. We then get K converging sequences of centers $(x_{i,\alpha})_\alpha$ and weights $(\mu_{i,\alpha})_\alpha$ such that $\mu_{i,\alpha} \to 0$ for $i = 1, \dots K$, and we can decompose

$$u_\alpha = u_\infty + \sum_{i=1}^{K} B_\alpha^i + \mathcal{R}_\alpha, \tag{4}$$

with $\mathcal{R}_\alpha \to 0$ in $H^2(M)$ as $\alpha \to \infty$. The B_α^i being bubbles that concentrate at their center $x_{i,\alpha}$, and whose expression is given in Section II.4.

The next goal is to prove that a version of this decomposition holds in stronger spaces and investigate its consequences. Defining $r_\alpha(x) := \min_{i=1,\dots K} d_g(x, x_{i,\alpha})$ the distance to the closest center and $\mu_\alpha := \max_{i=1,\dots K} \mu_{i,\alpha}$ the weight associated with the bubble that concentrates the slowest, we prove pointwise blow-up estimates. We obtain

$$\left|\nabla^j u_\alpha\right| \leq C \left(\mu_\alpha^{\frac{n-4}{2}} r_\alpha(x)^{4-n-j} + \|u_\infty\|_{L^\infty} \right) \qquad \text{for all } x \in M, \tag{5}$$

for $j = 0, 1, 2, 3$. Throughout the work, C will represent a positive constant whose exact value does not matter, and can change in-between lines. Proving (5) is a lot harder than proving (4), and (5) actually only holds for sequences of solutions while (4) is true for more general Palais-Smale sequences.

We will then use these bounds to prove stability results in Chapter IV, where we split our arguments between so-called low and high dimensions, and reach sharp conditions on the value that b can have for (1) to be stable. This way, we observe that in high dimensions $n \geq 9$, the main Equation will be stable unless

$$b = \frac{n^2 - 2n - 4}{2n(n-1)} S c_g \tag{6}$$

where Sc_g is the scalar curvature of the manifold, which is assumed to be locally conformally flat. On the other hand, when $n = 5$, the equation is stable if the mass of the operator $\Delta_g^2 + b\Delta_g + c$ is positive, as discussed in Section IV.1.

The Thesis is articulated as follows, note that more references for each part are provided at the beginning of each respective section. The first chapter consists in an introduction to the required notions in differential geometry an more specifically Riemannian geometry. In a second step, the theory of Sobolev spaces on smooth manifolds is also introduced. We refer to Hebey (1997); Hebey and Robert (2008). In the second chapter, we start by a motivation of the shape of the kind of equation we study, through conformal changes of metric. This part is inspired from S.-Y. A. Chang, Eastwood, Orsted, and Yang (2008). Then we develop the general theory for fourth-order equations, followed by important results regarding critical bi-Laplacian equations resembling the Paneitz one, this comes from Gilbarg and Trudinger (2001); Robert (2009). Finally, in the last section of this chapter, we discuss non-trivial results on the behaviour of Palais-Smale sequences for the energy functional. This part consists in an introduction to the phenomenon of concentration in bubbles, which is studied extensively in the rest of the work. This so-called H^2-theory for blow-up comes from Robert (2009); Hebey and Robert (2001). The third chapter in turn aims to provide estimates on the blow-up, to describe the pointwise concentration of solutions. It comes mainly from the first section of the main article of the Thesis, that is Hebey and Robert (2011). In the fourth and last chapter, we prove the main Theorems by analysing the asymptotical behavior of blowing-up sequences of solutions and ultimately discuss the stability in all dimensions. This last part comes from the second half of Hebey and Robert (2011), with additional results coming from Hebey, Robert, and Wen (2006).

Chapter I

Introduction to Riemannian geometry and Sobolev spaces

In this first introductory chapter, we present some basic tools and results that are used throughout the rest of the work. Starting with the definitions of important tools in the study of smooth manifolds and particularly of Riemannian geometry, we will discuss the notions without proving theorems. In a second step, we introduce Sobolev spaces on smooth compact manifolds and present their embeddings in classical space functions. This chapter comes from Hebey (1997), but the reader can also refer to Lee (2003, 2018) for the part on differential geometry and to Adams and Fournier (2003) for a more in-depth dive in Sobolev spaces in $\mathbb{R}^n$. The proofs of this chapter are omitted for the sake of brevity.

1 Differential geometry

The cornerstone of the theory relies on the concept of a smooth manifold of dimension n. We start with a topological space M, and we ask that for each point x, there exists a neighborhood U of x and an injective continuous map $\phi : U \to \mathbb{R}^n$, called *local chart* of M at x. One can then talk about *local coordinates* of $y \in U$ around some point x, the coordinates of y are the coordinates of $\phi(y)$ in $\mathbb{R}^n$.

Definition 1.1. A *smooth manifold* is a topological space M equipped with such a collection of charts $\{(U_i, \phi_i)\}$ called an *atlas* such that the resulting space is Hausdorff and second-countable, with the addition that we ask, when $U_i \cap U_j \neq \phi$, that the map

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \subset \mathbb{R}^n \to \phi_j(U_i \cap U_j) \subset \mathbb{R}^n$$

is of class $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

In the following, we will refer to manifold implying smooth manifolds, and also assume that they are connected spaces. Moreover, unless otherwise specified, all the manifolds will be orientable and considered to be without boundary.

Example 1.2.

i. As a trivial example, $\mathbb{R}^n$ is naturally endowed with a smooth n-dimensional manifold structure, with a unique global chart $(\mathbb{R}^n, \mathrm{Id})$.

ii. The unit sphere $S^n \subset \mathbb{R}^{n+1}$ can also be endowed with a smooth n-manifold structure. One way to do that is to define the stereographic projections, taking $U_N = S^n \setminus \{(0,\dots,0,-1)\}$ the sphere without the South Pole, and U_S similarly without the North Pole. Then we can define the stereographic projection through the North Pole π_N

$$\pi_N : U_N \to \mathbb{R}^n : x = (x_1, \dots x_{n+1}) \mapsto \left(\frac{x_1}{1+x_{n+1}}, \dots \frac{x_n}{1+x_{n+1}} \right) \tag{I.1}$$

and the stereographic projection through the South Pole

$$\pi_S : U_S \to \mathbb{R}^n : x = (x_1, \dots x_{n+1}) \mapsto \left(\frac{x_1}{1-x_{n+1}}, \dots \frac{x_n}{1-x_{n+1}} \right). \tag{I.2}$$

One can then verify that we have the inverse map, for instance in the northern hemisphere

$$\pi_N^{-1} : x \mapsto \left(\frac{2}{1+|x|^2}\, x, \frac{1-|x|^2}{1+|x|^2} \right) \in S^n. \tag{I.3}$$

The collection $\{(U_N, \pi_N), (U_S, \pi_S)\}$ is then an atlas on S^n.

With this definition, one can talk about differentiability of a function which takes it values in M, through the charts.

Definition 1.3. Let M be a n-manifold and N be a m-manifold, and $f : M \to N$. This function is said to be *differentiable* k times, $f \in C^k(M, N)$, if for all $x \in M$ and (U, ϕ) chart of M at x and (V, ψ) chart of N at $f(x)$, the map

$$\psi \circ f \circ \phi^{-1} : \phi(U) \subset \mathbb{R}^n \to \psi(V) \subset \mathbb{R}^m$$

is of class $C^k(\mathbb{R}^n, \mathbb{R}^m)$. If $N = \mathbb{R}$, we write $C^k(M)$ the set of k times differentiable functions $f : M \to \mathbb{R}$.

Differentiability allows to define the differential of a function at a point, which would be in comparison with the Euclidean case a linear map between two vector spaces. We then need the notion of a *tangent vector* at $x \in M$, $X : C^\infty(M) \to \mathbb{R}$, linear map such that

$$X(f.g) = f(x)X(g) + g(x)X(f). \tag{I.4}$$

This leads to the definition of the *tangent space* of M at $x \in M$, that is the space $T_xM := \{X : C^\infty(M) \to \mathbb{R}$ that are tangent vectors at $x\}$. Taking advantage of the linearity of the vectors, if (U, ϕ) is a chart of M at x with local coordinates $(x_1, \dots x_n)$, then a basis of T_xM is given by the collection of $\partial_i\big|_x$, defined as

$$\partial_i\big|_x(f) := \left(\frac{\partial}{\partial x_i}(f \circ \phi^{-1}) \right)(x) \tag{I.5}$$

where the partial derivative is the standard one on $\mathbb{R}^n$, and by simplicity we often talk about $\partial_i\big|_x(f)$ as the *partial derivative* of f at x written by abuse of notation $\frac{\partial f}{\partial x_i}(x)$. Then a general tangent vector at x is written $X = \sum_{i=1}^n X^i \partial_i\big|_x =: X^i \partial_i\big|_x$. From now on, a summation is implied every time an index is repeated both in superscript and subscript in an equation, this is the Einstein's summation convention.

Definition 1.4. Let M be a smooth manifold of dimension n, we define the *tangent bundle* of M, written TM, as the union of all tangent spaces, $TM := \bigcup_{x\in M} T_xM$.

Let $x \in M$ and a vector $X \in T_xM$, then taking a chart (U, ϕ) at x one can define a chart of TM at X with the map

$$\Phi : \bigcup_{x\in U} T_x(M) \to \mathbb{R}^n \times \mathbb{R}^n : X = X^i \partial_i\big|_y \mapsto (\phi(y) = y_1, \dots y_n, X^1, \dots X^n). \tag{I.6}$$

This endowes TM with a structure of smooth manifold of dimension $2n$, and T_xM is isomorphic to the euclidean space $\mathbb{R}^n$. We have now the necessary tools to define the differential of a function.

Definition 1.5. Let M, N be two manifolds, and $f : M \to N$ differentiable at least one time at x, we then define the *differential* of f at x as the linear map

$$df(x) : T_xM \to T_{f(x)}N : (df(x)X)(g) = X(g \circ f) \quad \forall\ g \in C^\infty(N).$$

When $N = \mathbb{R}$, and $f : M \to \mathbb{R}$, then the differential of f at x is a linear form on tangent vectors, $df : T_xM \to T_{f(x)}\mathbb{R} = \mathbb{R}$. In this case it is uniquely defined by the quantities $df(x)(\partial_i\big|_x) = \frac{\partial f}{\partial x_i}(x)$. This leads us to consider the dual basis of the $\partial_i\big|_x$, we write $dx^i\big|_x$ the linear maps such that

$$dx^i\big|_x(\partial_j\big|_x) = \delta^i_j \tag{I.7}$$

where $\delta^i_j = 1$ if $i = j$ and 0 otherwise. The $(dx^i\big|_x)_{i=1,\dots n}$ form a basis of the dual space of T_xM, denoted by T^*_xM, and we can express $df(x) = \partial_i f\ dx^i$. In general, all linear forms on tangent vectors are determined by the value they attribute to the basis, and can be uniquely decomposed $\eta = \eta_i dx^i$. The so-called *cotangent* bundle $T^*M = \bigcup_{x\in M} T^*_xM$ is also endowed with a $2n$-manifold structure through a similar chart map

$$\Phi(\eta = \eta_i dx^i\big|_x) = (x_1, \dots x_n, \eta_1, \dots \eta_n). \tag{I.8}$$

Now that we can talk about tangent vectors at a point, we would like to have a more global definition and consider vector fields on a manifold. Using the manifold structure of TM, we call X a C^k *vector field* on M if $X : M \to TM$ is of class $C^k(M, TM)$ such that $X(x) \in T_xM$, i.e. at each point $x \in M$ we associate a tangent vector at x with C^k regularity. Then $X(x) = X^i(x)\partial_i\big|_x$ in a local chart (U, ϕ) and all the X^i are $C^k(\phi(U))$ functions, in the same way we define *one-forms* as smooth fields in the cotangent bundle, $\eta(x) = \eta_i(x)dx^i\big|_x$ with $\eta_i \in C^\infty(\phi(U))$. We will write $\Gamma(M)$ the set of smooth vector fields on M and $\Omega^1(M)$ the set of one-forms on M.

Another central notion is the concept of a metric on a n-manifold M. Since T_xM is a vector space, we can endow it with a scalar product.

Definition 1.6. A *Riemannian metric* is a smooth *field* g defining a scalar product $g(x)$ on T_xM at each $x \in M$. In other words, we ask that if $X, Y \in \Gamma(M)$, $g(X, Y) : M \to \mathbb{R}$ is a smooth function and that for all $x \in M$, $g(x)$ is a bilinear positive-definite symmetric map on T_xM.

Fixing a point x and a chart at x with local coordinates $(x_1, \dots x_n)$, then a metric g is locally determined by the coefficients $g_{ij}(x) = g(x)(\partial_i, \partial_j)$. In the following we will often need $g^{ij} := (g^{-1})_{ij}$, as coefficients of the inverse matrix of the linear map, such that $g^{ij}g_{jk} = \delta^i_k$. The scalar product of two vector fields with local expression $X = X^i(x)\partial_i$, $Y = Y^i(x)\partial_i$ is then

$$g(x)(X, Y) = g_{ij}(x)X^i(x)Y^j(x). \tag{I.9}$$

Proposition 1.7 (Metric pullback). Let M, N two manifolds, and $f : M \to N$ an immersion, that is a smooth injective function whose differential map is injective too between $TM \to TN$. Assume g is a Riemannian metric on N, then the *pullback* of g through f, f^*g defined as

$$(f^*g)(x)(X,Y) := g(f(x))(df(X), df(Y)) \qquad \forall\ X, Y \in \Gamma(M),\ x \in M \tag{I.10}$$

is a Riemannian metric on M.

As we said, one can express the differential of a smooth function $f : M \to \mathbb{R}$ as $df = \partial_i f dx^i$, then the collection $(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n})$ forms a vector of $\mathbb{R}^n$, it is said to be *covariant*. In the same way, since a vector field can be decomposed as $X(x) = X^i(x)\partial_i$, the components $(X^1, \dots X^n)$ are said to be *contravariant* and one could ask whether differentiating the vector leads to a covariant object, that is to know if $(\partial_i X^j)_j$ can be the coefficients of a covariant vector. Actually, $\partial_i X = \frac{\partial X^j}{\partial x_i}\partial_j$ is no longer a global vector field, in differentiating the expression of X one needs also to derive the $\frac{\partial}{\partial x_i}$ term, and we define the *Christoffel symbols* Γ^k_{ij} as the k-th local coordinate in the basis (∂_i) of the partial derivative along x_i of the term x_j.

Definition 1.8. We define a *covariant derivative* at $x \in M$ as a linear map written $\nabla_i : \Gamma(M) \to \Gamma(M)$ such that for all $f \in C^\infty(M)$, $\nabla_i(fX) = \frac{\partial f}{\partial x_i}X(x) + f(x)\nabla_i X$.

In local coordinates the Christoffel symbols are defined as $\nabla_i\left(\frac{\partial}{\partial x_j}\right)(x) = \Gamma^k_{ij}(x)\frac{\partial}{\partial x^k}$ and a C^k vector field $X(x) = X^i(x)\partial_i$ is differentiated to get a C^{k-1} vector field

$$\nabla_i X = \left(\frac{\partial X^k}{\partial x_i} + \Gamma^k_{ij}X^j\right)\partial_k. \tag{I.11}$$

In the same way, for a smooth one-form $\eta(x) = \eta_i(x)dx^i$ the covariant derivative is expressed as the one-form

$$\nabla_i \eta = \left(\frac{\partial \eta_j}{\partial x_i} - \Gamma^k_{ij}\eta_k\right)dx^j. \tag{I.12}$$

For a smooth function $f : M \to \mathbb{R}$, we recover that the usual partial derivative is indeed a covariant one, $\nabla_i f = \partial_i f$.

We have already discussed some linear and multilinear forms on vectors, in general we call a (p,q)-*tensor* a smooth field of multilinear maps, at each $x \in M$

$$T(x) : (T_xM)^p \times (T^*_xM)^q \to \mathbb{R}, \tag{I.13}$$

such that if $X_1, \dots X_p$ are smooth vector fields and $\eta_1, \dots \eta_q$ are one-forms, the map

$$(T(X_1, \dots X_p, \eta_1, \dots \eta_q) : M \to \mathbb{R}) \in C^\infty(M). \tag{I.14}$$

In local coordinates, the tensor T is uniquely determined by its values on the basis elements,

$$T^{j_1,\dots j_q}_{i_1,\dots i_p}(x) = T(x)(\partial_{i_1}, \dots \partial_{i_p}, dx^{j_1}, \dots dx^{j_q}). \tag{I.15}$$

We can then express

$$T = T^{j_1,\dots j_q}_{i_1,\dots i_p}\ dx^{i_1} \otimes \dots dx^{i_p} \otimes \partial_{j_1} \otimes \dots \partial_{j_q} \tag{I.16}$$

taking the tensor product between the basis elements. With these considerations, one can apply a (p,q)-tensor on a single vector and get a $(p-1,q)$ tensor. For instance if A^{ij} are the local components of a symmetric $(0,2)$-tensor and writing $\nabla f = \nabla_i f dx^i$ the *gradient* of $f \in C^\infty(M)$, then

$$A(\nabla f) = A^{ij}\frac{\partial f}{\partial x_j}\partial_i \tag{I.17}$$

is a smooth vector field on M. Similarly, the Riemannian metric can be seen as a $(2,0)$-tensor with components g_{ij} and defines an isomorphism at each tangent space

$$g : T_xM \to T_x^*M : X = X^i\partial_i \mapsto g(X) = g_{ij}X^i dx^j. \tag{I.18}$$

Writing $X_i = g_{ij}X^i$, the metric allows to lower indices of contravariant objects, and to raise indices of covariant ones through $\eta^i = g^{ij}\eta_i$. This is called the *musical isomorphism*, sometimes noted $X^\flat = g(X)$, $\eta^\sharp = g^{-1}(\eta)$. In the same way, a general (p,q)-tensor has a covariant derivative that acts as in Equations (I.11,I.12) on each index

$$(\nabla_i T)^{j_1,\dots j_q}_{i_1,\dots i_p} = \frac{\partial T^{j_1,\dots j_q}_{i_1,\dots i_p}}{\partial x_i} - \sum_{l=1}^{p}\Gamma^\alpha_{ii_l}T^{j_1,\dots j_q}_{i_1,\dots i_{l-1},\alpha,i_{l+1},\dots i_p} + \sum_{l=1}^{q}\Gamma^{j_l}_{i\alpha}T^{j_1,\dots j_{l-1},\alpha,j_{l+1},\dots j_q}_{i_1,\dots i_p}. \tag{I.19}$$

Remark i : Note that in the following, we sometimes use the notation of ∇^k to express the higher orders derivatives and not to express a contravariant derivative where the index would have been raised. For instance

$$(\nabla^2 f)_{ij}(x) = (\nabla_i(\nabla f)(x))_j = \frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial f}{\partial x_k} \tag{I.20}$$

are the components of a $(2,0)$-tensor since ∇f is a one-form, and

$$\left|\nabla^2 f(x)\right|_g^2 = g^{ia}g^{jb}\ \nabla^2_{ab}f\ \nabla^2_{ij}f. \tag{I.21}$$

Similarly, $\nabla^3_{ijk}f = \partial_i(\nabla_{jk}f) - \Gamma^\alpha_{ij}\nabla_{\alpha k}f - \Gamma^\alpha_{ik}\nabla_{\alpha j}f$ and so on.

Remark ii : Note also that the gradient ∇f is a one-form, meaning it is a vector in the dual space of the tangent bundle. This means that one cannot rigorously make sense of notations as $g(\nabla f, \nabla h)$ with $f,h \in C^\infty(M)$ and should instead write $g^{-1}(\nabla f, \nabla h)$. In the following, the scalar product of two gradients in the dual space will sometimes be written

$$(\nabla f, \nabla h)_g := g^{-1}(\nabla f, \nabla h) = g^{ij}\partial_i f \partial_j h \tag{I.22}$$

in local coordinates. For simplicity, we sometimes forget to put the -1 exponent and in the following, each time a $(2,0)$ operator $S = S_{ij}dx^i \otimes dx^j$ is applied on a gradient we mean actually

$$\begin{aligned} S(\nabla f, \nabla h) &:= S^{\sharp\sharp}(\nabla f, \nabla h) = S^{ij}\partial_i f \partial_j h = g^{ia}g^{jb}S_{ab}\partial_i f \partial_j f \in \mathbb{R}. \\ S(\nabla f) &:= S^{\sharp\sharp}(\nabla f) = S^{ij}\partial_j f\ dx^i \in T^*M \end{aligned} \tag{I.23}$$

This way we write $|\nabla f|_g^2 = (\nabla f, \nabla f)_g$, and forget the g subscript when the Riemannian metric is obvious through context.

Whereas there exists an infinity of covariant derivatives on a smooth manifold, determined by the symbols Γ_{ij}^k also called symbols of the connection, in a Riemannian manifold one particular derivation arises.

Definition 1.9. Let (M, g) be a Riemannian manifold, we define the *Levi-Civita* connection as the unique covariant derivative such that the associated symbols have a symmetry in the indices $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all i, j, k and $\nabla_i g = 0$.

From now on, every time we will talk about a covariant derivative, it will refer to the Levi-Civita connection. By the definition, one must have in local coordinates $(\nabla_k g)_{ij} = (\nabla_i g)_{jk} = (\nabla_j g)_{ik} = 0$, i.e.

$$\begin{aligned} &\frac{\partial g_{ij}}{\partial x_k} - \Gamma_{kj}^l g_{il} - \Gamma_{ik}^l g_{lj} = 0, \\ &\frac{\partial g_{jk}}{\partial x_i} - \Gamma_{ik}^l g_{jl} - \Gamma_{ji}^l g_{lk} = 0, \\ &\frac{\partial g_{ik}}{\partial x_j} - \Gamma_{jk}^l g_{il} - \Gamma_{ij}^l g_{lk} = 0. \end{aligned} \tag{I.24}$$

Taking advantage of the symmetries on the indices, we get $\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k} = 2\Gamma_{ij}^l g_{kl}$, so that we have an explicit expression for the Christoffel symbols,

$$\Gamma_{ij}^k = \frac{1}{2}\left(\frac{\partial g_{lj}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right) g^{lk}. \tag{I.25}$$

With the ability to derive a vector field in a direction, one can try to follow a path along a smooth collection of tangent vectors at each point. Let $\gamma : [0, 1] \to M$ be a smooth path on M, at each point it has a tangent vector given by $\dot{\gamma}(t)$ defined as the linear map

$$\dot{\gamma}(t) : C^\infty(M) \to \mathbb{R} : f \mapsto \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}. \tag{I.26}$$

In a local chart, writing $\gamma(t) = (\gamma^1(t), \dots \gamma^n(t))$, we have $(\dot{\gamma}(t))^i = \frac{d\gamma^i}{dt}(t)$, we will write the time-derivatives with a dot.

Definition 1.10. We define a *geodesic* of M such a path γ such that it does not cause its tangent vector to bend, i.e. $(\nabla_{\dot{\gamma}})(\dot{\gamma}(t)) = 0$ for all $t \in [0, 1]$.

In local coordinates, a geodesic is a path that solves the ordinary differential equation

$$\ddot{\gamma}^k(t) + \Gamma_{ij}^k(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0, \qquad k = 1, \dots n, \tag{I.27}$$

for all t such that $\gamma(t)$ stays in the local chart. Now by existence of maximal solutions to this kind of ODE's, we know that at each point $x \in M$, and for all vector $X \in T_xM$ there exists $\varepsilon > 0$ and a unique geodesic $\gamma_x(X) : [0, \varepsilon] \to M$ such that

$$\gamma_x(X)(0) = x \qquad \text{and} \qquad \dot{\gamma}_x(X)(0) = X. \tag{I.28}$$

This corresponds to solving the differential equation with given initial conditions. Moreover, rescaling the starting vector corresponds to rescaling the time parameter

$$\begin{aligned} &X \rightsquigarrow \tilde{X} = \lambda X \\ &\gamma_x(X) \rightsquigarrow \tilde{\gamma}_x(\tilde{X}) : [0, \varepsilon/\lambda] \to M : t \mapsto \gamma_x(X)(t/\lambda). \end{aligned} \tag{I.29}$$

Up to taking λ small enough, one gets that there exists $r > 0$ such that for all $X \in T_xM$ with $|X|_g < r$, $\gamma_x(X)$ is defined up until $t = 1$, and this leads to the following.

Definition 1.11. We define the exponential at $x \in M$ as the map

$$\exp_x(X) : \{X \in T_xM \quad \text{s.t.} \ \ |X|_g = \sqrt{g(x)(X,X)} < r\} = B_0(r) \subset T_xM \to M$$

that sends a vector X to the value at $t = 1$ of the unique geodesic starting at x with tangent vector X, $\exp_x(tX) = \gamma_x(X)(t)$ for $t \in [0, 1]$.

With the natural isomorphism between T_xM and $\mathbb{R}^n$ given by $g(x)$, one can talk about the exponential of a vector in $\mathbb{R}^n$, and by the properties of the unique solution to an ordinary differential equation, $\exp_x \in C^\infty(B_0(r), M)$ defined on the Euclidean ball of radius r in $\mathbb{R}^n$, with

$$d\exp_x(0) = \mathrm{Id}_{\mathbb{R}^n \to T_xM}. \tag{I.30}$$

But this means that exp defines a local diffeomorphism between a neighborhood U_x of x in M and some neighborhood of 0 in $\mathbb{R}^n$, and we have a so-called *exponential chart* $(U_x, \exp_x^{-1})$, with the particularity that in the associated local coordinates $g(x)_{ij} = \delta_{ij}$ and $\Gamma^k_{ij}(x) = 0$ for all i, j, k. Moreover, when (M, g) is a compact manifold, then it must be geodesically complete (by the properties of the maximal solution) in the sense that all geodesics are defined at all times. If we define the distance between two points $x, y \in M$ as

$$d_g(x,y) = \inf_{\substack{\gamma \\ \gamma(0)=x, \gamma(1)=y}} \int_0^1 \sqrt{|\dot\gamma(t)|_g} dt, \tag{I.31}$$

locally a geodesic minimizes the distance. We write $\rho(x)$ the biggest positive number such that all geodesics starting at x with length $L(\gamma) < \rho(x)$ minimize the distance, this defines the *injectivity radius* of M, $i_g(M) = \inf_{x \in M} \rho(x)$. In other words, for all $x, y \in M$ such that $d_g(x, y) < i_g(M)$, we know that

$$d_g(x,y) = L(\gamma_{x\to y}) = L\begin{pmatrix} [0,1] & \to & M \\ t & \mapsto & \exp_x(tu) \end{pmatrix} \tag{I.32}$$

where $u \in \mathbb{R}^n$ is such that $\exp_x(u) = y$, and $d_g(x, y_\alpha) = \left|\exp_x^{-1}(y_a)\right| + o(1)$ for a sequence of converging points $y_\alpha \to x$ in M.

Geodesics on a manifold allow to transport a vector along without "changing its direction", but unlike in the Riemannian case, following a closed loop along geodesics does not necessarily take a vector to itself. This depends on the curvature of the space. The *Riemann curvature* tensor is the smooth $(3, 1)$-tensor defined in local coordinates as

$$(R_g(x))^l_{ijk} = \left(\frac{\partial \Gamma^l_{ik}}{\partial x_j}\right)(x) - \left(\frac{\partial \Gamma^l_{ij}}{\partial x_k}\right)(x) + \Gamma^l_{j\alpha}(x)\Gamma^\alpha_{ki}(x) - \Gamma^l_{k\alpha}(x)\Gamma^\alpha_{ji}(x). \tag{I.33}$$

This object quantifies uniquely the second-order geometry of the manifold, whereas Γ was a first-order quantity in g, and presents some symmetries in its indices. The unique non-zero contraction of two indices leads to the definition of the *Ricci curvature*, a $(2, 0)$-tensor, i.e. a bilinear map on tangent vectors, which is symmetric and such that in local coordinates $(Ricc_g(x))_{ij} = (R_g)^\alpha_{i\alpha j}$. Finally, the *scalar curvature* is the real valued function $Sc_g = g^{ij}(Ricc_g)_{ij} \in C^\infty(M)$, embedding the curvature information at each point in a single number.

On a Riemannian manifold, we also have a natural way to integrate real continuous function. We start with an atlas $(U_i, \phi_i)_i$, and a subordinated partition of unity $(\vartheta_j)_j$, that is a collection of smooth real valued compactly supported functions such that

i. $\forall\, x \in M, 0 \leq \vartheta_j \leq 1$;

ii. $\mathrm{supp}(\vartheta_j) \subset U_j$ for some chart domain U_j;

iii. At each $x \in M$, there is only a finite number of non-zero ϑ_j and $\sum_j \vartheta_j(x) = 1$.

This allows to localize a function $f : M \to \mathbb{R}$ in each particular chart, and we define the *Riemannian integral* of f as

$$\int_M f dv_g := \sum_j \int_{\phi_j(U_j)} \left(\vartheta_j \sqrt{|\det g|} f\right) \circ \phi_j^{-1} dx. \tag{I.34}$$

In the right-hand side the integration takes place in $\mathbb{R}^n$, and the expression $\det g$ is the determinant of the application's matrix in the local coordinates of U_j. The presence of the square root allows this definition to be independent of the choice of the atlas, and also the particular partition of unity. Note that dv_g is called the *volume form* of M, on an orientable manifold $dv_g = \sqrt{|\det g|} dx^1 \wedge \ldots dx^n$ is a n-form. To make sense of this definition, we would need a complete description of the theory of k-forms on manifolds, which is beyond the scope and needs of this Thesis. We then state some well-known facts about the integral.

Proposition 1.12 (Change of variable). Let $(M, g), (N, h)$ be two n-dimensional Riemannian manifold and $\varphi : M \to N$ a diffeomorphism (smooth bijective map with smooth inverse) that preserves the orientation of the tangent space, then for all $f \in C^\infty(M)$ with compact support we have

$$\int_N f \; dv_h = \int_M f \circ \varphi \;\; |\det d\varphi| \, dv_g. \tag{I.35}$$

By its expression, $\det d\varphi = \det(\frac{\partial \varphi_i}{\partial x_j})_{ij}$ and is sometimes called the *Jacobian* of φ.

We can also define some *differential operators*, that will be used throughout this work.

Definition 1.13. Let (M, g) be a smooth Riemannian manifold, and $f \in C^\infty(M)$ with gradient $\nabla f = \partial_i f dx^i$, $X \in \Gamma(M)$ a smooth vector field.

i. We define the *divergence* of a vector field, $\mathrm{div}_g : \Gamma(M) \to C^\infty(M)$ as the contraction of the covariant partial derivative, by analogy with the Euclidean case. In (I.11), we get that the gradient of a tangent vector can be seen as a $(1,1)$-tensor $\nabla X = (\nabla_i X)^j dx^i \otimes \partial_j$, its divergence is then

$$\mathrm{div}_g(X) := \mathrm{tr}(\nabla X) = \partial_i X^i + \Gamma^i_{ij} X^j = (\nabla_i X)^i \tag{I.36}$$

if $X = X^i \partial_i$ in local coordinates. Another equivalent and sometimes more useful expression is $\mathrm{div}_g(X^i \partial_i) = \frac{1}{\sqrt{|\det g|}} \frac{\partial}{\partial x_i} \left(\sqrt{|\det g|} X^i\right)$. One can also apply the divergence on a one-form

using the musical isomorphism and get finally in local coordinates, if $\eta = \eta_i dx^i$,

$$\begin{aligned}\operatorname{div}_g(\eta) &= \operatorname{div}_g(g^{ij}\eta_j\partial_i) = (\partial_i g^{ij})\eta_j + g^{ij}\partial_i\eta_j + \Gamma^i_{ij}g^{jk}\eta_k \\ &= g^{ij}\partial_i\eta_j - \Gamma^i_{ik}g^{kj}\eta_j - \Gamma^j_{ik}g^{ki}\eta_j + \Gamma^i_{ij}g^{jk}\eta_k \\ &= g^{ij}\left(\partial_i\eta_j - \Gamma^k_{ij}\eta_k\right) = g^{ij}(\nabla_i\eta)_j\end{aligned} \tag{I.37}$$

using that $(\nabla_i g)^{jk} = 0$.

ii. We define the *Laplacian* of a smooth function, also called Laplace-Beltrami operator, as $\Delta_g f := -\operatorname{div}_g(\nabla f) : C^\infty(M) \to C^\infty(M)$, in local coordinates

$$\Delta_g f = -g^{ij}(\nabla_i(\nabla f))_j = -g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial f}{\partial x_k}\right). \tag{I.38}$$

Using integration by parts, one then gets the divergence Theorem which states as follows.

Theorem 1.14. Let (M, g) be a smooth compact Riemannian manifold, and $f, h \in C^\infty(M)$, $X \in \Gamma(M)$ a smooth vector field, then

$$\begin{aligned}\int_M f \operatorname{div}_g(X) dv_g &= -\int_M df(X) dv_g = -\int_M (\nabla f, X)_g \, dv_g \\ \int_M \Delta_g f \; h \; dv_g &= -\int_M (\nabla f, \nabla h)_g \, dv_g = \int_M f \; \Delta_g h dv_g\end{aligned} \tag{I.39}$$

With an abuse of notation to see the gradient as a vector, in the first equation $(\nabla f, X)_g := g((\nabla f)^\sharp, X) = \left(\nabla f, X^\flat\right)_g = \partial_i f X^i$ in local coordinates.

This is a consequence of the more general Stokes theorem that will not be presented here.

2 Sobolev spaces on manifolds

Before discussing the main equations on manifolds that will be our principal interest, we still need to define some function spaces that are used throughout the work. This section focuses on these definitions, and we will also present some standard results about Sobolev spaces on manifolds, by drawing the parallel with the Euclidean setting. This section comes more specifically from Hebey and Robert (2008), and it is supposed that the reader already has some knoledge about the Sobolev space theory on domains of $\mathbb{R}^n$.

We can define the norms for $u \in C^\infty(M)$,

$$\begin{aligned}\|u\|_{C^0(M,g)} &= \|u\|_{L^\infty(M)} := (\text{ess}) \sup_{x\in M} |u(x)| \\ \|u\|_{C^k(M,g)} &:= \left\| |\nabla^k u|_g \right\|_{L^\infty(M)}, \qquad k \geq 1.\end{aligned} \tag{I.40}$$

Here and in the following we always have $\left|\nabla^k u\right|_g^2 := g^k(\nabla^k u, \nabla^k u) = g^{i_1,j_1} \dots g^{i_k,j_k} \nabla^k_{i_1,\dots i_k} u \nabla^k_{j_1,\dots j_k} u$. These norms give respectively to each $C^k(M)$ a structure of Banach space. We also introduce the

Lebesgue spaces $L^p(M,g)$, of functions on M with finite norm

$$\|u\|_{L^p(M,g)} := \left(\int_M |u|^p \, dv_g\right)^{1/p} < \infty, \tag{I.41}$$

these are also Banach spaces, and specifically, $L^2(M)$ is a Hilbert space with the scalar product $(u,v)_{L^2} := \int_M u\, v\, dv_g$ for all $u, v \in L^2(M)$.

The spaces we will introduce naturally arise when trying to solve partial differential equations on manifolds, indeed the integration by part theorem allows to define the partial derivative of a function in the weak sense by testing it against particular functions.

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^n$, and $u \in L^1_{loc}(\Omega)$, the *distributional derivative* of order $\alpha = (\alpha_1, \dots \alpha_n)$ is $\partial_\alpha u \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega \partial_\alpha u \; \varphi dx = (-1)^{|\alpha|} \int_\Omega u \; \partial_\alpha \varphi dx \qquad \forall \; \varphi \in C_0^\infty(\Omega). \tag{I.42}$$

Here $C_0^\infty(\Omega)$ stands for the set of all smooth compactly supported functions in Ω so that the partial derivative in the right-hand side is the standard one on $\mathbb{R}^n$.

This allows to define the first *Sobolev space*,

$$H^{k,p}(\Omega) := \{u \in L^p(\Omega) \quad \text{s.t. for } |\alpha| \le k, \; \partial_\alpha u \in L^p(\Omega) \text{ exists as in (I.42)}\}. \tag{I.43}$$

Giving it the norm

$$\|u\|_{H^{k,p}} := \left(\sum_{0 \le |\alpha| \le k} \|\partial_\alpha u\|^p_{L^p(\Omega)}\right)^{1/p}, \tag{I.44}$$

$H^{k,p}(\Omega)$ results to be a Banach space. We have the following result, giving an equivalent definition of the space.

Theorem 2.2 (Meyers-Serrin). Let $\Omega \subset \mathbb{R}^n$, an integer $k \ge 0$ and $p \ge 1$, then

$$H^{k,p}(\Omega) = \overline{C^\infty(\Omega)}^{\|\cdot\|_{H^{k,p}}} \tag{I.45}$$

the completion in L^p of the set of smooth functions for the norm defined in (I.44).

This Theorem leads to a very natural generalization on smooth manifolds.

Definition 2.3. Let (M,g) be a Riemannian manifold, an integer $k \ge 0$ and $p \ge 1$, we define the set $\mathcal{C}^{k,p}(M,g) := \{u \in C^\infty(M) \quad \text{s.t. } \int_M \left|\nabla^j u\right|^p_g dv_g < \infty \text{ for } j = 0, \dots k\}$. Then we define the *Sobolev space* of order (k,p), $H^{k,p}(M,g)$, as the completion of $\mathcal{C}^{k,p}(M,g)$ under the norm

$$\|u\|_{H^{k,p}} := \left(\sum_{j=0}^k \int_M \left|\nabla^j u\right|^p_g dv_g\right)^{1/p}. \tag{I.46}$$

Remark i : As in the Euclidean case one can try to define the weak derivative of a function using the integration by parts. Sobolev spaces are exactly well suited spaces for this. Indeed, if $u \in H^{k,p}(M,g)$

with $k \geq 1$, we can define its weak differential through the local diffeomorphism with $\mathbb{R}^n$. Let $x \in M$, and take a chart (U, ϕ) at x, then $u \circ \phi^{-1} \in H^{k,p}(\phi(U))$ and admits weak partial derivatives. We define $du(x)$ through

$$du(x)(X) = \partial_k (u \circ \phi^{-1})(\phi(x)) X^k \qquad \forall\ X = X^k \partial_k\big|_x \in T_x M \tag{I.47}$$

with the derivative in the right-hand side defined almost everywhere in $\phi(U)$.This definition does not depend on the choice of the chart at x, and then the gradient of u is defined through the duality coming from g, $du(x)(X) = g(x)(X, \nabla u)$.

Remark ii : All the spaces $H^{k,p}(M, g)$ are Banach spaces with their respective norm, and moreover $H^{k,2}(M, g) =: H^k(M, g)$ is a Hilbert space for all integers $k \geq 0$, with the scalar product

$$(u, v)_{H^k} = \sum_{j=0}^{k} \int_M \left(\nabla^j u, \nabla^j v\right)_g dv_g. \tag{I.48}$$

Example 2.4. Let (M, g) be a smooth compact Riemannian manifold, and $u \in H^{1,p}(M)$. Then $|u| \in H^{1,p}(M)$ and

$$|\nabla |u||_g (x) = |\nabla u|_g (x) \tag{I.49}$$

for almost every $x \in M$.

With the help of weak derivatives, one can define *weak solutions* to a differential problem. Let $u \in H^{1,p}(M, g)$, u is a weak solution to the equation $\Delta_g u = f$ with $f \in L^p(M)$ if

$$\int_M (\nabla u, \nabla \varphi)_g \, dv_g = -\int_M f\ \varphi\ dv_g \qquad \forall \varphi \in C_0^\infty(M). \tag{I.50}$$

Similarly, if A is a smooth symmetric bi-linear map on tangent vectors, then $u \in H^{1,p}(M, g)$ is a weak solution to the equation $-\operatorname{div}_g(A(\nabla u)) = f$ if

$$\int_M A(\nabla u, \nabla \varphi) dv_g = \int_M f\ \varphi\ dv_g \qquad \forall \varphi \in C_0^\infty(M). \tag{I.51}$$

When M is a compact smooth manifold, $C_0^\infty(M) = C^\infty(M)$ and thus by continuity and density of this set, definitions such as (I.50,I.51) must hold true taking $\varphi \in H^1(M, g)$.

It seems that the space $H^{k,p}(M, g)$ is highly dependent on the choice of the metric, since it changes the norm of the derivatives of a function in the integral. However, when M is a compact manifold, then it can be covered by a finite collection of open domains of charts, in which one can find $c > 0$ such that $c^{-1} g \leq g' \leq cg$ for any two metrics g, g' on M. Since M is covered by a finite amount of such domains, there is a global constant c and g, g' are equivalent in the sense of scalar products. This means that when M is compact, $H^{k,p}(M, g) = H^{k,p}(M, g')$ and we can talk about $H^{k,p}(M)$ without ambiguity. This is one of the reasons we will always consider M to be compact in the following. Another one will be the following embedding Theorems extending the Sobolev embedding Theorem in the Euclidean setting.

Theorem 2.5 (Sobolev embedding 1st part). Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$, $p \geq 1$ and an integer $k \geq 0$ such that $k < \frac{n}{p}$. Then $H^{k,p}(M) \hookrightarrow L^q(M)$ with $q = \frac{np}{n-kp}$ in the sense that $H^{k,p}(M) \subset L^q(M)$ and the embedding is continuous,

$$\|u\|_{L^q} \leq C \|u\|_{H^{k,p}} \tag{I.52}$$

for some constant $C > 0$.

Note that in the case of a compact manifold we have the continous embeddings $L^p(M) \subset L^q(M)$ whenever $p \geq q$, so that $H^{k,p}(M) \subset H^{k,q}(M)$. We also have, if $m \leq k$, $H^{k,p}(M) \subset H^{m,p}(M)$.

In the non-compact case, one could think that since the Theorem holds on $\mathbb{R}^n$ it would also be valid on a general n-manifold. However, the geometry of the manifold plays an important role in the validity of such results, and there are counter-examples of manifolds where no Sobolev embedding holds. We present here the following necessary condition (see Hebey & Robert, 2008, Proposition 4.2) as an example, even though in the rest of this Thesis we always work in compact smooth manifolds, connected, without boundaries and orientable.

Proposition 2.6. Let (M, g) be a geodesically complete manifold of dimension n. Let us assume that the Sobolev embedding holds true for k, p such that $kp < n$, that is $H^{k,p}(M, g) \hookrightarrow L^q(M, g)$ with $q = \frac{np}{n-kp}$ as above. Then there exists a constant $c > 0$ such that for all $r > 0$,

$$\mathrm{Vol}_g(B_x(r)) \geq c \min\{1, r^n\} \tag{I.53}$$

for all $x \in M$.

In the following we will need also intermediary regularity spaces, for $\alpha > 0$, $C^{k,\alpha}(M)$ is defined as the set of C^k functions such that the ratio

$$\frac{|u(x) - u(y)|}{d_g(x, y)^\alpha} \tag{I.54}$$

stays bounded for all $x \neq y$. Endowing this space with the norm

$$\|u\|_{C^{k,\alpha}} := \|u\|_{C^k} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{d_g(x, y)^\alpha}, \tag{I.55}$$

it has a structure of Banach space.

Theorem 2.7 (Sobolev embedding 2nd part). Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$, $p \geq 1$ and an integer $k \geq 0$. If $k > \frac{n}{p}$, then take $0 < \alpha \leq 1$ such that one can write $k - \frac{n}{p} = m + \alpha$ for an integer $m \geq 0$. We have $H^{k,p}(M) \hookrightarrow C^{m,\alpha}(M)$. In particular there is a constant $C > 0$ with

$$\|u\|_{C^{m,\alpha}} \leq C \|u\|_{H^{k,p}}. \tag{I.56}$$

The following theorem uses all the properties of the several inclusions we discussed and specifies in which way the embedding occurs in the sub-critical case.

Theorem 2.8 (Compactness of the embedding). Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$. Let $p \geq 1$ and an integer $k \geq 1$ such that $k < \frac{n}{p}$. Taking $m < k$ and $1 \leq q \leq \frac{np}{n-(k-m)p} =: q_k$, we have $H^{k,p}(M) \hookrightarrow H^{m,q}(M)$. Moreover if $q < q_k$, the embedding is compact.

In the case $p = 2$, $k = 1$, we will write $2^* := \frac{2n}{n-2}$ and when $p = 2$, $k = 2$, we write $2^\# := \frac{2n}{n-4}$ the *critical exponents* for the compactness of the embeddings $H^k \hookrightarrow L^q$, $k = 1, 2$.

Chapter II

Bi-Laplacian equations

1 Derivation through conformal transformations

In this section, we will motivate the choice of our main equation by making it appear in a context of conformal transformations of Riemannian quantities. Conformal transformations of smooth manifolds through diffeomorphisms are important concepts in Riemannian geometry. They have been extensively studied in the last century both by mathematicians and physicists. The interested reader is for instance referred to Obata (1970), Schoen and Yau (1979) or Schoen (1984) for earlier works on the matter, while the structure and the definitions of this section come from S.-Y. A. Chang, Eastwood, Orsted, and Yang (2008), one can also refer to S. Chang and Yang (2008); S.-Y. A. Chang (2005).

Definition 1.1. Let (M, g) be a smooth compact Riemannian manifold of dimension n, a metric $\hat{g}$ is said to be *conformal* to g if there is a function $f \in C^\infty(M)$ such that $\hat{g} = e^{2f}g$. One can then talk about *conformally covariant* operators of bidegree (a, b), that is operators A depending on the metric that transform as

$$A_{\hat{g}}(\varphi) = e^{-bf} A_g(e^{af}\varphi) \qquad \forall\ \varphi \in C^\infty(M). \tag{II.1}$$

Exploring several dimensions. For instance, in dimension $n = 2$, one can easily verify that the Laplacian transforms conformally with bidegree $(0, 2)$, since

$$\Delta_{e^{2f}g} = e^{-2f}\Delta_g. \tag{II.2}$$

A notion of conformal curvature is naturally associated to this operator. By Gauss-Bonet Theorem, we know that on a closed compact manifold

$$\int_M Sc_g dv_g = 4\pi\chi(M) \tag{II.3}$$

where $\chi(M)$ is the Euler characteristic of the manifold. The latter is invariant, so that we know that Sc_g must also transform conformally, and in fact $Sc_{\hat{g}} = e^{-2f}(Sc_g + 2\Delta_g f)$ expressed only with covariant terms, recalling $\Delta_g = -\operatorname{div}_g(\nabla)$. And indeed we get the invariance of the integral using the divergence Theorem.

The equation linking two scalar curvatures in a conformal change of the metric leads to the well known problem of classification of surfaces. The aim is to find for any metric g on a smooth compact two-dimensional manifold, a conformal representative that has constant scalar curvature of value -1, 0 or 1. This amounts to finding $f \in C^\infty(M)$ such that

$$\Delta_g f + \frac{1}{2} Sc_g = \frac{1}{2} Sc_{\hat{g}} e^{2f}, \tag{II.4}$$

with $Sc_{\hat{g}} = 1, 0, -1$ depending on the sign of the Euler characteristic.

When going to dimensions greater than two, one must change the Laplacian expression a little to get a conformal operator. This leads to the definition of *conformal Laplacian*, for a compact smooth manifold of dimension $n \geq 3$,

$$L_g = \Delta_g + \frac{n-2}{4(n-1)} Sc_g. \tag{II.5}$$

Then under the conformal change of metric $\hat{g} = \upsilon^{\frac{4}{n-2}} g$ that will interest us later on (setting $e^f = \upsilon^{\frac{2}{n-2}}$), one gets

$$L_{\hat{g}}(\varphi) = \upsilon^{-\frac{n+2}{n-2}} L_g(\upsilon\varphi), \tag{II.6}$$

i.e. L_g transforms conformally with bidegree $(\frac{n-2}{2}, \frac{n+2}{2})$. The scalar curvature is still a covariant quantity, and $Sc_g, Sc_{\hat{g}}$ are linked by the *Yamabe equation*, taking $\varphi \equiv 1$ in (II.6)

$$L_g \upsilon = \Delta_g \upsilon + \frac{n-2}{4(n-1)} Sc_g \upsilon = \frac{n-2}{4(n-1)} Sc_{\hat{g}} \upsilon^{\frac{n+2}{n-2}}. \tag{II.7}$$

As in the two-dimensional case, we can try to solve this Equation and find υ such that $Sc_{\hat{g}} = Cst$. This is the so-called Yamabe problem, that has been solved by the work of several mathematicians such as Yamabe, Trudinger, Aubin and Schoen, and which will be further discussed in this chapter later on.

Going to higher order operators, it is natural to ask if such conformal equations still exist. In a lecture of 1983 that was published posthumously, Paneitz (2008) proposed such an operator of fourth-order, that transforms conformally. In dimension $n = 4$, we define the fourth-order curvature called *Q-curvature*

$$Q_g := \left(\frac{1}{6} \Delta_g Sc_g - \frac{1}{2} |Ricc_g|^2 + \frac{1}{6} Sc_g^2 \right) \tag{II.8}$$

where $|Ricc_g|^2 = (Ricc_g)^{ab}(Ricc_g)_{ab}$ using the Einstein summation convention to contract the Ricci tensor. Then this quantity transforms in a similar way to the two-dimensional curvature, under the conformal change of metric $\hat{g} = e^{2f} g$,

$$Q_{\hat{g}} = e^{-4f}(Q_g + Pan_g f) \tag{II.9}$$

where Pan_g is the Paneitz-Branson operator in dimension four,

$$Pan_g : C^\infty(M) \to C^\infty(M) : Pan_g(f) = \Delta_g^2 f - \mathrm{div}_g \left(\left(\frac{2}{3} Sc_g \; g - 2Ricc_g \right) (\nabla f) \right). \tag{II.10}$$

In dimension $n = 4$, this operator transforms conformally with bidegree $(4, 0)$,

$$Pan_{e^{2f}g} = e^{-4f} Pan_g. \tag{II.11}$$

A motivation to call Q a curvature comes from the fact that on one hand it transforms in a similar way to the standard scalar curvature, and on the other hand we have a similar invariant than (II.3). On a locally conformally flat closed 4-manifold,

$$\int_M Q_g dv_g = 8\pi^2 \chi(M), \tag{II.12}$$

and in a more general closed manifold, additional curvature terms appear in the integral.

This construction can then be extended in arbitrary even degrees and dimensions $2m$, by getting an operator with leading term Δ_g^m, of the form

$$Lu = (\Delta_g)^m u + \sum_{k=0}^{m-1} (-1)^k \nabla^k_{j_1 \dots j_k} \big((A_k)^{i_1 \dots i_k, j_1 \dots j_k} \nabla^k_{i_1 \dots i_k} u \big), \quad u \in C^\infty(M), \tag{II.13}$$

where A_k denotes a $(0, 2k)$-tensor field in M. The existence of such operators which transform well under conformal changes of the metric is proved by Graham, Jenne, Mason, and Sparling (1992), and are since then referred as GJMS operators.

On the other hand, one can extend the definition of Q-curvature to dimensions $n \geq 5$ (see for instance Robert, 2009).

Definition 1.2. Let (M, g) be a smooth compact manifold of dimension $n \geq 5$, then we define the Q-curvature

$$Q_g = \frac{1}{2(n-1)} \Delta_g Sc_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} Sc_g^2 - \frac{2}{(n-2)^2} |Ricc_g|^2$$

and the Paneitz-Branson operator

$$Pan_g u = \Delta_g^2 u - \operatorname{div}_g \left((a_n Sc_g \, g + b_n Ricc_g)(\nabla u) \right) + \frac{n-4}{2} Q_g u$$

with

$$a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)} \qquad b_n = -\frac{4}{n-2}.$$

When $n = 4$ we get back the previously defined quantities. This operator has bidegree $(\frac{n-4}{2}, \frac{n+4}{2})$, under a conformal change of metric $\hat{g} = v^{\frac{4}{n-4}} g$,

$$Pan_{\hat{g}}(\varphi) = v^{-\frac{n+4}{n-4}} Pan_g(v\varphi) \qquad \forall \, \varphi \in C^\infty(M), \tag{II.14}$$

and $Q_{\hat{g}}, Q_g$ are linked by a similar Equation

$$Pan_g v = \frac{n-4}{2} Q_{\hat{g}} v^{\frac{n+4}{n-4}}. \tag{II.15}$$

Now as in the case of the conformal Laplacian, one can try to find a conformal metric that has constant Q-curvature. This leads to the problem that will be our main focus in the following, we want to find $u \in C^\infty(M)$ such that $Q_{\hat{g}} = Cst$, that is

$$Pan_g u = \Delta_g^2 u - \mathrm{div}_g\left(A_g(\nabla u)\right) + hu = Cu^{\frac{n+4}{n-4}} = Cu^{2^\#-1} \tag{II.16}$$

where $A_g = (a_n Sc_g\ g + b_n Ricc_g)$ is a smooth symmetric bi-linear map on vectors of M applied to ∇u with abuse of notation, and $h \in C^\infty(M)$. As in the Yamabe problem, this is a critical equation in the sense of the Sobolev embeddings, with $2^\# = \frac{2n}{n-4}$ and will be discussed in depth later on.

Example 1.3.

i. On $(\mathbb{R}^4, \xi)$, $\Delta_\xi = -\Delta$ and $Pan_\xi = \Delta^2$, the interesting kind of equations to solve would be

$$\Delta^2 u = u^{2^\#-1}. \tag{II.17}$$

All positive solutions to this equation are known (see Lin, 1998), as we will discuss later on.

ii. On a general manifold (M, g) of dimension $n \geq 5$ with an Einstein metric, that is $Ricc_g = \lambda g$ for some $\lambda \in \mathbb{R}$, then we can compute

$$Ricc_g = \frac{1}{n} Sc_g\ g \qquad |Ricc_g|^2 = \lambda^2 n.$$

The equation to solve is

$$Pan_g u = \Delta_g^2 u + b_n \lambda \Delta_g u + c_n \lambda^2 u = \frac{n-4}{2} Q_{\hat{g}} u^{2^\#-1} \tag{II.18}$$

with

$$b_n = \frac{n^2 - 2n - 4}{2(n-1)} \qquad c_n = \frac{n(n-4)(n^2-4)}{16(n-1)^2}.$$

In particular, we get $b_n^2 - 4c_n = \frac{4}{(n-1)^2} > 0$.

Example 1.4. On (S^n, h), where h is the restriction of the standard Euclidean metric on $\mathbb{R}^{n+1}$ to the surface of the sphere, then $Ricc_h = \frac{1}{n} Sc_h\ h$, $Sc_h = n(n-1)$ and thus for instance for $n = 4$, $Pan_h = \Delta^2 - 2\Delta$.

We already know that (S^n, h) solves the Yamabe problem and the Paneitz equation since it has constant scalar curvature and Q-curvature. We also know that there is a local diffeomorphism from S^n to $\mathbb{R}^n$ through the stereographic projection π. Pulling the sphere's metric back into $\mathbb{R}^n$ through the inverse of the stereographic projection we then get a new conformal metric on the Euclidean space,

$$(\pi^{-1})^* h = \frac{4}{(1+|x|^2)^2} \xi_{\mathbb{R}^n} \qquad \text{for any } x \in \mathbb{R}^n. \tag{II.19}$$

This means that in each chart h is conformally equivalent to the Euclidean metric ξ, the sphere is said to be *locally conformally flat*. Concerning the second-order case, let υ be such that $(\pi^{-1})^* h = \upsilon^{\frac{4}{n-2}} \xi$, thus

$$\upsilon = 2^{\frac{n-2}{2}} \left(1 + |x|^2\right)^{-\frac{n-2}{2}}. \tag{II.20}$$

Then υ solves

$$L_\xi \upsilon = -\Delta \upsilon = \frac{n(n-2)}{4} \upsilon^{\frac{n+2}{n-2}} \qquad \text{in } \mathbb{R}^n. \tag{II.21}$$

Respectively in fourth order, we let

$$\upsilon = 2^{\frac{n-4}{2}} \left(1 + |x|^2\right)^{-\frac{n-4}{2}}, \qquad \text{so that} \quad \upsilon^{\frac{4}{n-4}} = \frac{4}{(1+|x|^2)^2}. \tag{II.22}$$

Then υ solves, in $\mathbb{R}^n$,

$$Pan_\xi \upsilon = \Delta^2 \upsilon = C_n \, \upsilon^{\frac{n+4}{n-4}}. \tag{II.23}$$

This geometric example shows the importance of positive solutions of (II.16) in $\mathbb{R}^n$. They will have a central role as we will see from the Section 4 on, as these functions locally appear as bubbles for our problem.

2 Elliptic theory

In the previous section, we introduced the type of equation we will study in the following, that is equations of the form

$$\Delta_g^2 u - \operatorname{div}_g(A_g \nabla u) + au = f u^{2^\# - 1}, \tag{II.24}$$

where A_g is a smooth symmetric bi-linear map and $a, f \in C^\infty(M)$ to be as general as possible. Before diving more closely on the results of existence, uniqueness and regularity of solutions to this kind of problem, we start this section by stating some important properties for elliptic equations of fourth order. In particular, we will often go back and forth between second-order equations and fourth-order ones for the similarity of the results. The reader is referred to Han and Lin (2011); Gilbarg and Trudinger (2001) for further exploration of the theory and the proof of the results presented here.

Definition 2.1. Let L be a linear differential operator of order m on a compact manifold (M, g) of dimension n, given by

$$Lu = \sum_{1 \le |\alpha| \le m} a_\alpha(x) \nabla_\alpha u + d(x) u, \tag{II.25}$$

where $\alpha = (\alpha_1, \alpha_2, \dots \alpha_n)$ is a multi-index. The operator L is said to be *elliptic* if for every $x \in M$ and non-zero $\xi \in \mathbb{R}^n$,

$$\sum_{|a|=m} a_\alpha(x) \xi^\alpha \neq 0, \tag{II.26}$$

where $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_n^{\alpha_n}$.

This definition can then be extended to non-linear operator whose leading-order term is linear, since the definition only depends on the higher order terms. It can also be defined for general non-linear operators, saying that L is elliptic if its linearization is.

These kinds of operators are of great interest since they are the basis many physical and mathematical models, it is then natural to try and solve so-called Dirichlet problems for the above mentioned operators. On the Euclidean space, let $\Omega \subset \mathbb{R}^n$ be a domain and L an elliptic operator on Ω with bounded coefficients, f a function on Ω and φ a function on $\partial\Omega$, the Dirichlet problem is to find u with sufficient regularity such that

$$\begin{cases} Lu = f & \text{on } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{II.27}$$

Regularity and existence. In this work we will only be interested in elliptic operators of even degree, in particular of degree 2 and 4, since they present regularity results that will be very useful for the following. Indeed, in this latter case, if L is of degree $2m$ with bounded coefficients, and $f \in L^2(\Omega)$, $\varphi \in H^m(\Omega)$, solving (II.27) in the weak sense amounts to finding $u \in H^m(\Omega)$ such that $u - \varphi \in H_0^m(\Omega)$ and

$$\int_\Omega Lu\ v\ dx = \int_\Omega f\ v dx \qquad \forall\ v \in C_0^\infty(\Omega), \tag{II.28}$$

making sense of the derivatives in the left-hand side by partial integration.

Theorem 2.2 (Interior regularity). Let $u \in H^m(\Omega)$ be a weak solution of the equation $Lu = f$ in Ω where the coefficients a_α are uniformly Lipschitz continuous in Ω, and $d \in L^\infty(\Omega)$, $f \in L^2(\Omega)$. Then for all subdomain Ω' whose closure is contained in Ω, we write $\Omega' \subset\subset \Omega$, $u \in H^{2m}(\Omega')$ and there is a constant depending on Ω' and the coefficients of L such that

$$\|u\|_{H^{2m}(\Omega')} \leq C\left(\|u\|_{H^m(\Omega)} + \|f\|_{L^2(\Omega)}\right). \tag{II.29}$$

In fact one can gain at most $2m$ differentiability, that is if $u \in H^m(\Omega)$ is a weak solution of $Lu = f$ in Ω, where L has coefficients a_α of regularity $C^{k+1}(\overline{\Omega})$, and $f \in H^k(\Omega)$ for $k \geq 1$, then for any subdomain $\Omega' \subset\subset \Omega$ we have $u \in H^{k+2m}(\Omega')$ and

$$\|u\|_{H^{k+2m}(\Omega')} \leq C\left(\|u\|_{H^1(\Omega)} + \|f\|_{H^k(\Omega)}\right). \tag{II.30}$$

This result can easily be extended to our problem. In the case of a compact manifold M of dimension n, then no boundary value is needed to specify the problem and the weak formulation of a general fourth-order equation of the shape (II.24) is the following. Let A be a smooth symmetric bi-linear map on vectors of M and $a \in C^\infty(M)$, and $f \in L^2(M)$. We say that $u \in H^2(M)$ is a weak solution of $P_g u = \Delta_g^2 u - \mathrm{div}_g(A\nabla u) + au = f$ if for all $v \in C_0^\infty(M)$,

$$\int_M P_g u\ v dv_g = \int_M \left(\Delta_g u \Delta_g v + A(\nabla u, \nabla v) + auv\right) dv_g = \int_M f v dv_g. \tag{II.31}$$

The interior regularity extends in this case to global regularity, and we have the following (Robert, 2009).

Theorem 2.3 (Elliptic regularity). Let (M,g) be a smooth compact Riemannian manifold of dimension n, let $a \in C^\infty(M)$ and A a smooth bi-linear map, and $f \in H^k(M)$ for $k \geq 0$. If $u \in H^2(M)$ is a weak solution to $P_g u = f$ in M, then $u \in H^{k+4}(M)$ and we have a constant depending on M, g and the norms of a, A such that

$$\|u\|_{H^{k+4}(M)} \leq C \left(\|f\|_{H^k(M)} + \|u\|_{L^2} \right). \tag{II.32}$$

Theorem 2.4 (Schauder theory). Let (M,g) be a smooth compact Riemannian manifold of dimension n, let $a \in C^\infty(M)$ and A a smooth bi-linear map. If $f \in C^{k,\alpha}(M)$ for some $\alpha > 0$ and $u \in H^2(M)$ is a weak solution to $P_g u = f$ in M, then $u \in C^{k+4,\alpha}(M)$ is a classical solution, and we have

$$\|u\|_{C^{k+4}(M)} \leq C \left(\|f\|_{C^{k,\alpha}(M)} + \|u\|_{C^k(M)} \right). \tag{II.33}$$

These so-called *a priori* bounds all rely on the existence of a weak solution to the problem.

Definition 2.5. Let P_g defined as in (II.50), it is said to be *coercive* if there exists $\lambda > 0$ such that

$$\int_M \left((\Delta_g u)^2 + A(\nabla u, \nabla u) + a u^2 \right) dv_g \geq \lambda \int_M u^2 dv_g$$

for all $u \in H^2(M)$.

Coercivity leads naturally to existence results, based on the very general Lax-Milgram Theorem.

Proposition 2.6. Let (M,g) be a smooth compact Riemannian manifold, A be a symmetric bi-linear map on vectors of M and $a \in C^\infty(M)$. Assuming the operator $P_g = \Delta_g^2 - \mathrm{div}_g(\nabla \cdot) + a$ is coercive, then for all $f \in H^{k,p}(M)$, $1 < p < \infty$, there is a unique $u \in H^{k+4,p}(M)$ such that $P_g u = f$, and since P_g is coercive we have now

$$\|u\|_{H^{k+4,p}(M)} \leq C \|f\|_{H^{k,p}(M)}. \tag{II.34}$$

In the second-order setting, we have the following existence result, that comes from Gilbarg and Trudinger (2001, Theorem 8.3).

Proposition 2.7. Let (M,g) be a smooth compact Riemannian manifold, $a \in C^\infty(M)$. Defining a coercive operator $L = \Delta_g + a$, in particular L is uniformly elliptic on M and the problem $Lu = f$ is uniquely solvable for all $f \in L^2(M)$.

Moreover, in degree two the uniform ellipticity of the operator leads to a maximum principle on domains of $\mathbb{R}^n$.

Theorem 2.8 (Strong maximum principle). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and define the second-order operator $Lu = \Delta_g u + au$. Then if $a \geq 0$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is such that $Lu \leq 0$ in Ω (respectively $Lu \geq 0$), u cannot achieve its maximum (respectively minimum) in the interior of Ω unless it is constant. In other words either

$$u \equiv C \quad \text{on } \Omega \qquad \text{or} \qquad u(x) < \max_{y \in \partial\Omega} u(y) \quad \text{for all } x \in \Omega \setminus \partial\Omega. \tag{II.35}$$

And this ensures the positivity of the solutions when f is positive on manifolds.

Theorem 2.9. Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$, and $a \in C^\infty(M)$. If $L = \Delta_g + a$ is coercive on H^1, that is there is $\lambda > 0$ such that

$$\int_M \left(|\nabla u|^2 + au^2 \right) dv_g \geq \lambda \int_M u^2 dv_g, \tag{II.36}$$

then for all $u \in C^2(M)$ such that $\Delta_g u + au \geq 0$, either $u \equiv 0$ or $u > 0$ on M.

Proof. Suppose we have $u \in C^2(M)$ such that $Lu \geq 0$. We define $u_- := \max(-u, 0) \geq 0$, then $u_- \in H^1(M)$ and $\nabla u_- = -\mathbb{1}_{\{u \leq 0\}} \nabla u$ as in Example I.2.4. This gives us

$$\begin{aligned} 0 \leq \int_M u_- \, L u dv_g &= \int_M \left((\nabla u, \nabla u_-)_g + a u u_- \right) dv_g \\ &\leq - \int_M \left(|\nabla u_-|^2 + a u_-^2 \right) dv_g \leq -\lambda \left\| u_- \right\|_{L^2}^2 . \end{aligned} \tag{II.37}$$

Thus $u_- \equiv 0$ and $u \geq 0$. But then $\Delta_g u + (a + \|a\|_\infty) u \geq \|a\|_\infty u \geq 0$, $a + \|a\|_\infty \geq 0$ and so by strong maximum principle either $u > 0$ or $u \equiv 0$ on M. □

Green functions. The existence of a solution to a simple bi-Laplacian equation has been studied extensively. Let $\Omega \subset \mathbb{R}^n$ be a domain, if we look at the bi-Laplacian boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{on } \Omega \\ u = |\nabla u| = 0 & \text{on } \partial\Omega, \end{cases} \tag{II.38}$$

and we assume that the domain Ω is regular enough, for every $f \in C^{0,\alpha}(\overline{\Omega})$, $\alpha > 0$, one can find a unique $u \in C^{4,\alpha}(\overline{\Omega})$ that is solution to the problem. This defines an operator called the *Green function* G_Ω, corresponding to the problem (II.38). It is a $L^1(\Omega \times \Omega)$-function defined on the product space with the diagonal removed, such that

$$u(x) = \int_\Omega G_\Omega(x, y) f(y) dy \qquad \text{for all} x \in M. \tag{II.39}$$

This last equality is called a *representation formula*.

Several questions can then be asked on this function, in particular investigate whether $G_\Omega(x, y) \geq 0$ or even $G_\Omega(x, y) > 0$ for $x, y \in \Omega$, as discussed in Grunau and Sweers (2007). This corresponds to ask : if $f > 0$ on Ω, can we say that the corresponding solution to the boundary problem $u > 0$ in Ω? Whereas in the second-order setting this is answered by the Theorem 2.9, in fourth order this is no longer true for all domains, as we will discuss in the next section. Another important question is to get estimates on the behavior of G_Ω, particularly for close x and y for which the Green function becomes singular. We have the following result coming from the shape of the fundamental solution to the bi-Laplacian on $\mathbb{R}^n$.

Proposition 2.10. Let $n \geq 5$, and $\Omega \subset \mathbb{R}^n$, then there exists a constant depending on the dimension, $\delta_n > 0$ such that, writing $d(x) := \mathrm{dist}_g(x, \partial\Omega)$, if $x \neq y \in \Omega$ with $|x-y| < \delta_n \max(d(x), d(y))$ then

$$0 < G_\Omega(x,y) \leq C \begin{cases} |x-y|^{4-n} & \text{if } n \geq 5, \\ \log\left(1 + \frac{1}{|x-y|}\right) & \text{for } n = 4 \end{cases} \tag{II.40}$$

where $C > 0$ is a constant depending on the domain.

Whereas the construction of the Green function for the bi-Laplacian requires a little more care on a smooth manifold, in the case of a coercive operator the Proposition 2.6 allows us to define the Green's function for P_g as a perturbation of the bi-Laplacian. As in domains of $\mathbb{R}^n$ discussed in Grunau and Robert (2010), we have the following estimates on the Green function.

Proposition 2.11. Let (M, g) be a smooth compact manifold of dimension $n \geq 5$. Let $P_g = \Delta_g^2 + b\Delta + c$ be a coercive bi-Laplacian operator with $C^\infty(M)$-coefficients. Then there is a unique Green function $G \in L^1(M \times M)$ defined everywhere except on the diagonal such that for all $\varphi \in C^4(M)$ one has the representation formula

$$\varphi(x) = \int_M G(x,y) \left(\Delta_g^2 \varphi(y) + b(y)\Delta_g\varphi(y) + c(y)\right) dv_g \qquad \text{for all } x \in M. \tag{II.41}$$

The following estimate holds for all $x \neq y \in M$,

$$|G(x,y)| \leq C d_g(x,y)^{4-n}. \tag{II.42}$$

Moreover, we have for all $x \in M$, writing $G_x = G(x, \cdot)$, that $G_x \in L^1(M) \cap C^4(M \setminus \{x\})$, and then

$$\nabla^k G_x(y) \leq C d_g(x,y)^{4-n-k} \qquad \text{for all} x \neq y, \tag{II.43}$$

for $k = 0, 1, 2, 3$.

Finally, we consider the constant-coefficients operator $P_g = \Delta_g^2 + b\Delta_g + c$, and assume that $b^2 - 4c > 0$ so that it can be decomposed as $P_g = (\Delta_g + d_1) \circ (\Delta_g + d_2)$, writing

$$d_1 := \frac{b}{2} + \sqrt{\frac{b^2}{4} - c}, \qquad d_2 := \frac{b}{2} - \sqrt{\frac{b^2}{4} - c}. \tag{II.44}$$

The latter models the Paneitz-Branson operator in an Einstein metric, in which case the bi-Laplacian problem $P_g u = f$ can be decomposed in a system of two coupled equations

$$\begin{cases} \Delta_g v + d_1 v = f \\ \Delta_g u + d_2 u = v \end{cases} \qquad \text{on } M. \tag{II.45}$$

In the case that $\Delta_g + d_1$, $\Delta_g + d_2$ and P_g are coercive, then the uniqueness of the solutions implies the uniqueness of the Green function. Then it follows from the representation formulas that, denoting G the Green function of P_g on M, H_1, H_2 the Green functions of $\Delta_g + d_1$ and $\Delta_g + d_2$ respectively,

$$G(x,y) = \int_M H_1(x,z) H_2(z,y) dv_g(z). \tag{II.46}$$

And we have the same kind of estimates on H as in (II.42),

$$|H_i(x,y)| \leq C d_g(x,y)^{2-n} \tag{II.47}$$

for all $x \neq y \in M$. The convergence of the integral (II.46) and the estimate on G are provided by the following general result, referred to as Giraud's lemma in the article of Hebey and Robert (2011), that can be found in (Hebey, 2014, Lemma 7.5).

Lemma 2.12. Let (M,g) be a compact manifold of dimension n, X, Y two functions defined on $M \times M$ except on the diagonal. Assume there is $C > 0$, $\alpha, \beta > 0$ such that

$$|X(x,y)| \leq C d_g(x,y)^{\alpha-n} \quad |Y(x,y)| \leq C d_g(x,y)^{\beta-n} \tag{II.48}$$

for all $x \neq y$, with $\alpha, \beta < n$ so that X and Y are integrable.
Define $Z(x,y) := \int_M X(x,z)Y(z,y)dv_g(z)$, then Z is continuous on $M \times M$ except on the diagonal and

$$|Z(x,y)| \leq \begin{cases} C d_g(x,y)^{\alpha+\beta-n} & \text{if } \alpha+\beta < n \\ C(1+|\log(d_g(x,y))|) & \text{if } \alpha+\beta = n \\ C & \text{if } \alpha+\beta > n \end{cases} \tag{II.49}$$

for some constant C that is independent of $x \neq y$. Moreover, if $\alpha + \beta > n$, Z is continuous on the whole $M \times M$.

3 Analysis of the problem

In this section the main focus will be to study general equations involving a bi-Laplacian and a critical non-linear term, and in particular pointing out what are the main difficulties we face when determining a positive solution. We will start by discussing the regularity and the positivity of a given weak solution, and comparing those discussions to the easier second degree. This section comes from Robert (2009).

We start with a general critical bi-Laplacian equation, on (M,g) a smooth Riemannian manifold of dimension $n \geq 5$, let A be a symmetric bi-linear map on vectors of M and $a, f \in C^\infty(M)$. Solving our problem consists in finding $u \in C^4(M)$ such that

$$\Delta_g^2 u - \operatorname{div}_g(A(\nabla u)) + au = f u^{2^\# - 1} \qquad \text{with } u > 0 \text{ in M.} \tag{II.50}$$

We will write $P_g := \Delta_g^2 - \operatorname{div}_g(A(\nabla \cdot)) + a$, we want to know first whether there exists any solutions at all, and then if the set of those solutions is compact in $C^4(M)$-topology.

Definition 3.1. Let us define $R_f : H^2(M) \setminus \{0\} \to \mathbb{R}$ by

$$R_f(u) := \frac{\int_M \left((\Delta_g u)^2 + A(\nabla u, \nabla u) + au^2\right) dv_g}{\left(\int_M f\,|u|^{2^\#}\, dv_g\right)^{2/2^\#}},$$

called the *Rayleigh quotient* of the operator P_g. It is well-defined when $f > 0$ since the embedding of $H^2 \hookrightarrow L^{2^\#}$ is continuous.

One can check that R_f is differentiable and that

$$DR_f(u)\varphi = \frac{2}{\left(\int_M f\,|u|^{2^\#}\,dv_g\right)^{2/2^\#}} \int_M \left(\Delta_g u \Delta_g \varphi + A(\nabla u, \nabla\varphi) + au\varphi - \lambda(u) f\,|u|^{2^\#-2}\,u\varphi\right) dv_g$$
$$\text{for all } \varphi \in H^2(M), \text{ with } \lambda(u) := \frac{\int_M uP_g u dv_g}{\int_M f\,|u|^{2^\#}\,dv_g}. \tag{II.51}$$

This means that if u is solution to our problem (II.50), then $\tilde{u} := (\lambda(u))^{-\frac{1}{2^\#-2}}$ has $DR_f(\tilde{u}) = 0$. And conversely if $\tilde{u}$ is a critical point of the Rayleigh quotient, $u := \lambda(\tilde{u})^{\frac{1}{2^\#-2}}\tilde{u}$ will be a weak solution to $P_g u = f\,|u|^{2^\#-2}\,u$.

Solving the initial problem therefore comes down to finding critical points such as minimizers for the Rayleigh quotient, that is, find $u \in H^2(M) \setminus \{0\}$ such that

$$R_f(u) = \inf_{v \in H^2(M)\setminus\{0\}} R_f(v). \tag{II.52}$$

This will give us a weak solution $u \in H^2(M) \setminus \{0\}$ solving our equation $P_g u = f\,|u|^{2^\#-2}\,u$. We will then need to prove that we can choose u such that $u \geq 0$ almost everywhere in M. Finally, we will prove a regularity result to obtain solution $u \in C^4(M)$ such that $u > 0$ in M and u solves $P_g u = f u^{2^\#-1}$. To achieve each and everyone of these steps we will need conditions to impose on the operator P_g to be able to solve the problem.

Definition 3.2. We start by defining a new norm on $H^2(M)$, from now on we will write

$$\|u\|^2_{H^2(M)} = \|\Delta_g u\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2} \quad \forall\, u \in H^2(M).$$

This norm is better suited to study our problem and it is actually equivalent to the standard H^2-norm $\left(\sum_{j=0}^2 \left\|\nabla^j \cdot\right\|^2_{L^2}\right)^{1/2}$.

Regularity. The first situation we will analyse is the regularity of the solutions, assuming they exist. In the sub-critical case, the regularity will easily result from a bootstrap argument, and the proof will only be presented for completeness.

Proposition 3.3. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, let A be a symmetric bi-linear map on vectors of M and $a, f \in C^\infty(M)$ such that $f > 0$. Let $u \in H^2(M)$ be a weak solution to the equation

$$\Delta_g^2 u - \operatorname{div}_g\left(A(\nabla u)\right) + au = f\,|u|^{q-2}\,u \tag{II.53}$$

with $2 \leq q < 2^\#$. Then $u \in C^{4,\alpha}(M)$ for some $\alpha > 0$ and thus is a standard solution to the Equation.

Proof. We start with $u \in H^2(M) \hookrightarrow L^{2^\#}(M)$, then $P_g u = f\,|u|^{q-2}\,u \in L^{\frac{2^\#}{q-1}}$. Thus by the regularity Theorem, $u \in H^{4,\frac{2^\#}{q-1}}(M)$, and there could be three cases.

– If $\frac{4}{n} > \frac{q-1}{2^\#}$, then Sobolev's embedding Theorem gives us that $u \in C^{0,\alpha}(M)$ for some $\alpha > 0$, and thus $P_g u \in C^{0,\alpha}(M)$. Using once again the regularity Theorem, we get the conclusion $u \in C^{4,\alpha}(M)$.

– Else, if $\frac{4}{n} = \frac{q-1}{2^\#}$, then Sobolev's embedding Theorem gives us that $u \in L^p(M)$ for all $p \geq 1$. The regularity Theorem tell us that $u \in H^{4,p'}$ for all $p' \geq 1$. Choosing p' big enough so that $n < 4p'$, we have $u \in C^{0,\alpha}(M)$ by Sobolev's embedding for some $\alpha > 0$, and then as in the first case we get the conclusion $u \in C^4(M)$.

– Finally, if $\frac{4}{n} < \frac{q-1}{2^\#}$, then Sobolev's embedding Theorem gives us that $u \in L^{q_1}(M)$, with

$$q_1 = \frac{\frac{2^\#}{q-1}\, n}{n - 4\, \frac{2^\#}{q-1}}. \tag{II.54}$$

And we come back to the three possibles outcomes, either $\frac{4}{n} > \frac{q-1}{q_1}$, or $=$ or $<$. In the worst case we end up with a sequence of q_i, with

$$q_{i+1} = \frac{\frac{q_i}{q-1}\, n}{n - 4\, \frac{q_i}{q-1}}, \tag{II.55}$$

but this sequence tends to infinity for $q_0 = 2^\#$ and $q < 2^\#$, thus no matter the $2 \leq q < 2^\#$, the process must end after a finite amount of steps. We always end up with $u \in C^{4,\alpha}(M)$ for some positive α.

□

Remark i : First, we notice that the proof above relies on the fact that $q < 2^\#$, and thus it does not work in the critical case, for which $q_1 = 2^\#$ and we do not gain integrability at each step. However, what really starts the bootstrap procedure and ends up chaining Sobolev's embeddings and regularity Theorem to achieve $u \in C^4(M)$, is the fact that we always ended up with $u \in L^p(M)$ with $p > 2^\#$. If we can prove that this holds for some $p > 2^\#$ even in the critical case, then the same procedure will work and we will be able to conclude that u is a standard solution.

Remark ii : We also notice that since all the embeddings are continuous, and that regularity Theorem gives us bounds on u depending on the norm of P_g, the bootstrap provides the following estimate. For $u \in H^2(M)$ weak solution to (II.53), there is $C > 0$ depending on $q < 2^\#$ such that

$$\|u\|_{C^4(M)} \leq C \|u\|_{H^2(M)}. \tag{II.56}$$

In the critical case, we will still be able to gain the regularity of a weak solution to $P_g u = f\,|u|^{2^\#-2}\, u$, but such estimates on its C^4-norm will not exist.

Theorem 3.4. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, let A be a symmetric bi-linear map on vectors of M and $a, f \in C^\infty(M)$ such that $f > 0$. Let $u \in H^2(M)$ be a weak solution (not necessarily positive) to the equation (II.50), then $u \in C^{4,\alpha}(M)$ for some $\alpha > 0$.

Proof. As we just mentioned, we only need to find some $p > 2^\#$ such that $u \in L^p(M)$, then applying the bootstrap procedure we will get the required regularity. Let us start by assuming that

P_g is a coercive operator, as defined in 2.5. We will split u in two parts, in the sets $\{|u| \leq R\}$ and $\{|u| \geq R\}$ with $R > 0$ that will be chosen later on.

Since by Sobolev's embedding $u \in L^{2^\#}(M)$, $f\,|u|^{2^\#-2} \in L^{\frac{n}{4}}(M)$. Let $v \in L^p(M)$ be any function, with $p \geq 1$ fixed. Then $f\,|u|^{2^\#-2}\,\mathbb{1}_{\{|u|\geq R\}} v \in L^r(M)$ with $\frac{1}{r} = \frac{4}{n} + \frac{1}{p}$ by Hölder's inequality, and

$$\left\| f\,|u|^{2^\#-2}\,\mathbb{1}_{\{|u|\geq R\}} v \right\|_{L^r} \leq \|f\|_\infty \left\| |u|^{2^\#-2}\,\mathbb{1}_{\{|u|\geq R\}} \right\|_{L^{n/4}} \|v\|_{L^p}\,. \tag{II.57}$$

Since P_g is coercive, from regularity theory there is a unique $w \in H^{4,r}(M)$ such that

$$\begin{aligned} &P_g w = f\,|u|^{2^\#-2}\,\mathbb{1}_{\{|u|\geq R\}} v, \text{and} \\ &\|w\|_{H^{4,r}} \leq C\,\|f\|_\infty \left\| |u|^{2^\#-2}\,\mathbb{1}_{\{|u|\geq R\}} \right\|_{L^{n/4}} \|v\|_{L^p}\,. \end{aligned} \tag{II.58}$$

From its definition we have $4 \leq n/r$, so Sobolev's embedding gives us $H^{4,r}(M) \hookrightarrow L^q(M)$ for $q \leq \frac{nr}{n-4r} = p$. Thus, the operator mapping v to $T_{p,R}(v) = w$ defined as the unique solution to $P_g w = f\,|u|^{2^\#-2}\,\mathbb{1}_{\{|u|\geq R\}} v$ is well-defined, and

$$\begin{aligned} &T_{p,R} : L^p(M) \to L^p(M) \text{ with} \\ &\|T_{p,R}\|_{L^p\to L^p} \leq C\,\|f\|_\infty \left\| |u|^{2^\#-2}\,\mathbb{1}_{\{|u|\geq R\}} \right\|_{L^{n/4}} = C\,\|f\|_\infty \left(\int_{|u|\geq R} |u|^{2^\#}\,dv_g \right)^{4/n}. \end{aligned} \tag{II.59}$$

Since $u \in L^{2^\#}(M)$, we can find R_0 large enough so that $\|T_{p,R_0}\|_{L^p\to L^p} \leq \frac{1}{2}$, and thus $\mathrm{Id}_{L^p} - T_{p,R_0}$ is a continuous linear map with continuous inverse.

Independently, $f\,|u|^{2^\#-2}\,u\mathbb{1}_{\{|u|\leq R\}} \in L^\infty(M)$, so for all $p \geq 1$ there exists $u_R \in H^{4,p}(M)$ such that

$$P_g u_R = f\,|u|^{2^\#-2}\,u\mathbb{1}_{\{|u|\leq R\}}. \tag{II.60}$$

Now by hypothesis u is a weak solution to (II.50), so

$$\begin{aligned} &P_g u = f\,|u|^{2^\#-2}\,u = f\,|u|^{2^\#-2}\,u\mathbb{1}_{\{|u|\leq R\}} + f\,|u|^{2^\#-2}\,u\mathbb{1}_{\{|u|\geq R\}} \\ \Leftrightarrow\ &P_g(u - u_R) = f\,|u|^{2^\#-2}\,u\mathbb{1}_{\{|u|\geq R\}} \end{aligned} \tag{II.61}$$

in other words $T_{2^\#,R_0}(u) = u - u_R$, since $u \in L^{2^\#}(M)$. Let $p \geq 2^\#$, rewriting the latter we get $u_R = \left(\mathrm{Id}_{L^{2^\#}} - T_{2^\#,R_0}\right) u$, we then take $\overline{u} \in L^p(M)$ defined by $\overline{u} = \left(\mathrm{Id}_{L^p} - T_{p,R_0}\right)^{-1} u_R$ since the application is invertible. But since we took $p \geq 2^\#$,

$$\begin{aligned} \left(\mathrm{Id}_{L^p} - T_{p,R_0}\right)\overline{u} &= u_R \\ &= \overline{u} - T_{p,R_0}(\overline{u}), \end{aligned} \tag{II.62}$$

by definition of T, that means that

$$P_g(\overline{u} - u_R) = f\,|u|^{2^\#-2}\,\mathbb{1}_{\{|u|\geq R\}}\overline{u}. \tag{II.63}$$

But $\overline{u} - u_R \in L^{2^\#}(M)$, so that we also have $u_R = \left(\mathrm{Id}_{L^{2^\#}} - T_{2^\#,R_0}\right)\overline{u}$. Finally,

$$\left(\mathrm{Id}_{L^{2^\#}} - T_{2^\#,R_0}\right) u = u_R = \left(\mathrm{Id}_{L^p} - T_{p,R_0}\right)\overline{u} = \left(\mathrm{Id}_{L^{2^\#}} - T_{2^\#,R_0}\right)\overline{u}, \tag{II.64}$$

by uniqueness of the inverse, $u \equiv \overline{u} \in L^p(M)$, and we can take any $p > 2^\#$ to begin the bootstrap.

In the case that P_g is not coercive, there may not be a unique inverse defining T, so we take $K > 0$ such that $P_g + K$ is coercive. As before we define

$$\begin{aligned} &\tilde{T}_{p,R} : L^p(M) \to L^p(M) \quad \text{such that } \tilde{T}_{p,R}(v) = w \quad \text{if} \\ &(P_g + K)w = f\,|u|^{2^\#-2}\,\mathbb{1}_{\{|u|\geq R\}}v + K\ v\mathbb{1}_{\{|u|\geq R\}}. \end{aligned} \tag{II.65}$$

Then we have $\tilde{u}_R \in H^{4,p}(M)$ solution to

$$(P_g + K)\tilde{u}_R = f\,|u|^{2^\#-2}\,\mathbb{1}_{\{|u|\leq R\}}u + K\ u\mathbb{1}_{\{|u|\leq R\}}, \tag{II.66}$$

by hypothesis

$$\begin{aligned} (P_g + K)u &= f\,|u|^{2^\#-2}\,u + Ku \\ &= (P_g + K)\tilde{u}_R + f\,|u|^{2^\#-2}\,u\mathbb{1}_{\{|u|\geq R\}} + K\ u\mathbb{1}_{\{|u|\geq R\}}. \end{aligned} \tag{II.67}$$

With the same arguments, for a R_0 large enough to get the inversibility we define

$$\begin{aligned} &\overline{u} = \left(\mathrm{Id}_{L^p} - \tilde{T}_{p,R_0}\right)^{-1}\tilde{u}_R \quad \in L^p(M), \quad \text{then} \\ &\tilde{u}_R = \left(\mathrm{Id}_{L^{2^\#}} - \tilde{T}_{2^\#,R_0}\right)u = \left(\mathrm{Id}_{L^{2^\#}} - \tilde{T}_{2^\#,R_0}\right)\overline{u}. \end{aligned} \tag{II.68}$$

And this concludes the proof. □

Positivity. Assume we have some weak solution $u \in H^2(M)$, can we in some way use it to find a non-negative solution to our equation (II.50), and then can we make sure this solution is strictly positive?

Coming back to second-order Yamabe-type equations, this question is easily answered. Indeed, if $u \in H^1(M)$ is a weak solution to

$$\Delta_g u + au = f\,|u|^{\frac{4}{n-2}}\,u, \tag{II.69}$$

it leads to a critical point

$$\tilde{u} = \left(\frac{\int_M |\nabla u|^2 + au^2 dv_g}{\int_M f\,|u|^{\frac{2n}{n-2}}\,dv_g}\right)^{-\frac{n-2}{4}} \tag{II.70}$$

of the corresponding Rayleigh quotient

$$R^2_f(u) := \frac{\int_M \left(|\nabla u|^2 + au^2\right)dv_g}{\left(\int_M f\,|u|^{\frac{2n}{n-2}}\,dv_g\right)^{\frac{n-2}{n}}}. \tag{II.71}$$

But then since $|\tilde{u}| \in H^1(M)$, and that $|\nabla\,|u|| = |\nabla u|$, $R^2_f(|\tilde{u}|) = R^2_f(\tilde{u})$, we get that $|\tilde{u}|$ is also a critical point. Coming back to the equation, up to multiplying $|\tilde{u}|$ by a constant, that gives a positive $u \in H^1(M)$ which is a weak solution to the equation (II.69). Using a bootstrap argument, we also have in degree two that $u \in C^2(M)$ (see Hebey, 2014).

Let us hereby consider the fourth-order setting and observe that the strategy immediately fails since $u \in H^2(M)$ does not imply that $|u| \in H^2(M)$. Take for example $u(x) = x_i \in C^\infty(\mathbb{R}^n) \subset H^2_{loc}(\mathbb{R}^n)$ on a bounded domain Ω. Then in the distributional sense, for $\varphi \in C^\infty_0(\mathbb{R}^n)$,

$$\begin{aligned}\langle \Delta |u|, \varphi\rangle &:= \int_{\mathbb{R}^n} |x_i| \Delta\varphi(x) dx^n = \int_{x_i\geq 0} x_i \Delta\varphi(x) dx^n - \int_{x_i\leq 0} x_i \Delta\varphi(x) dx^n \\ &= \int_{\partial\{x_i>0\}} x_i \nabla\varphi d\sigma - \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \frac{\partial\varphi}{\partial x_i} dx_i\right) dx^{n-1} \\ &\qquad - \int_{\partial\{x_i<0\}} x_i \nabla\varphi d\sigma + \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^0 \frac{\partial\varphi}{\partial x_i} dx_i\right) dx^{n-1} \\ &= 2\int_{x_i=0} \varphi(x) dx^{n-1}\end{aligned} \tag{II.72}$$

which cannot be represented by some $v \in L^1_{loc}$ as $\int_{\mathbb{R}^n} v\varphi dx^n$. So $|u|$ does not have a weak Laplacian, and is not in $H^2_{loc}(\mathbb{R}^n)$.

Moreover, in degree two as seen in Theorem 2.9, coercivity of the operator and existence of a comparison principle were two equivalent conditions. In solving Yamabe's problem, we had achieved that $u \geq 0$ was a weak solution to (II.69), with $u \in C^2(M)$. Even in the case that $\Delta_g + a$ would not be coercive, adding a large enough K such that $\Delta_g u + (a - fu^{\frac{n+2}{n-2}} + K)$ is coercive, then

$$\Delta_g u + (a - fu^{\frac{n+2}{n-2}} + K)u = Ku \geq 0. \tag{II.73}$$

That allows us to use the strong comparison principle, Theorem 2.9, we get $u > 0$ that solves the initial problem. An important difference for fourth-order equations is that coercivity no longer implies a comparison principle.

Definition 3.5. In the fourth-order setting, we say that the bi-Laplacian operator P_g satisfies the pointwise comparison principle if for any $u \in C^4(M)$ such that $P_g u \geq 0$, then either $u > 0$ or $u \equiv 0$ in M.

If we impose the coercivity and that P_g verifies the pointwise comparison principle, we do achieve the positivity of the solution, solving the problem. Note that without these assumptions, we could have a Rayleigh quotient R_f whose only minimizers have to change sign.

Theorem 3.6. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, let A be a symmetric bi-linear map on vectors of M and $a, f \in C^\infty(M)$ such that $f > 0$. Assuming that P_g is coercive and that it verifies the pointwises comparison principle, then if $u \in H^2(M)$ is a minimizer of R_f, $u \in C^4(M)$ and either $u > 0$ or $u < 0$.

Proof. The regularity follows from the Theorem 3.4, and if u is a non-zero minimizer of R_f, we know there is $\mu \in \mathbb{R}$ such that $P_g u = \mu f |u|^{2^\# - 2} u$. Since P_g is coercive, there also is $\lambda > 0$ such that

$$\begin{aligned}&\int_M u P_g u dv_g \geq \lambda \|u\|^2_{L^{2^\#}} \quad \forall\ u \in H^2(M),\ \text{so} \\ &R_f(u) = \frac{\int_M u P_g u\ dv_g}{\left(\int_M f |u|^{2^\#} dv_g\right)^{2/2^\#}} \geq \frac{\lambda}{\|f\|^{2/2^\#}_\infty},\end{aligned} \tag{II.74}$$

this gives $\mu > 0$. We then use the coercivity of P_g to obtain a unique $v \in H^2(M)$ such that $P_g v = |P_g u|$ in M, and since $u \in C^{4,\alpha}(M)$ for some $\alpha > 0$, $v \in C^{4,\alpha}(M)$. It is easily seen that

$$\begin{cases} P_g v + P_g u = |P_g u| + P_g u \geq 0 \\ P_g v - P_g u = |P_g u| - P_g u \geq 0. \end{cases} \tag{II.75}$$

But we assumed that P_g satisfies the pointwise comparison principle, so $P_g(v \pm u) \geq 0$ implies $v \pm u \geq 0$ in M. In other words, $v \geq |u|$, and in particular $v \not\equiv 0$ since $\mu > 0$. By comparison principle once again one gets $v > 0$ in M.

We compute

$$\begin{aligned} R_f(v) = \frac{\int_M v P_g v \, dv_g}{\left(\int_M f\,|v|^{2^\#} dv_g\right)^{2/2^\#}} &= \mu \frac{\int_M v f\,|u|^{2^\#-1} dv_g}{\left(\int_M f v^{2^\#}\right)^{2/2^\#}} \\ &\leq \mu \frac{\left(\int_M f v^{2^\#} dv_g\right)^{\frac{1}{2^\#}} \left(\int_M f\,|u|^{2^\#} dv_g\right)^{1-\frac{1}{2^\#}}}{\left(\int_M f v^{2^\#} dv_g\right)^{\frac{2}{2^\#}}} && \text{(by Hölder's inequality)} \\ &\leq \mu \frac{\left(\int_M f\,|u|^{2^\#}\right)^{\frac{2^\#-1}{2^\#}}}{\left(\int_M f v^{2^\#} dv_g\right)^{\frac{1}{2^\#}}} \leq \mu \left(\int_M f\,|u|^{2^\#} dv_g\right)^{\frac{2^\#-2}{2^\#}} && \text{(since } v \geq |u|\text{)} \\ &\leq \inf_v R_f(v) && \text{(by the definition of } \mu\text{).} \end{aligned} \tag{II.76}$$

This means that v also reaches the minimum of R_f, so all the inequalities become equalities and we have $|u| = v > 0$. Then, since u is continuous on M either $u > 0$ or $u < 0$ everywhere. □

Now, we would like to know in which cases do a bi-Laplacian operator satisfy the pointwise comparison principle, and especially if the Paneitz operator belongs to that class.

Lemma 3.7. Let $a_1, a_2 \in C^\infty(M)$, such that the operators $\Delta_g + a_1$ and $\Delta_g + a_2$ are coercive on H^1, as defined in Theorem 2.9, then the product operator $P_g := (\Delta_g + a_1) \circ (\Delta_g + a_2)$ satisfies the pointwise comparison principle.

Proof. Let $u \in C^4(M)$ be such that $P_g u = (\Delta_g + a_1) \circ (\Delta_g + a_2)\, u \geq 0$, then by Theorem 2.9, either $(\Delta_g + a_2)\, u \equiv 0$, and $u \equiv 0$ on M since $(\Delta_g + a_2)$ is coercive, or $(\Delta_g + a_2)\, u \geq 0$, so that $u > 0$ or $u \equiv 0$ by coercivity once again. □

Actually, we have a necessary and sufficient condition on the operator so that $P_g = \Delta_g^2 - \mathrm{div}_g\,(A(\nabla\,\cdot\,) + a$ can be written as a product of two second-degree Laplacian operators.

Theorem 3.8. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, let A be a symmetric bi-linear map on vectors of M and $a \in C^\infty(M)$. Then $P_g = \Delta_g^2 - \mathrm{div}_g\,(A(\nabla\,\cdot\,) + a$ can be expressed as $P_g u = (\Delta_g + a_1) \circ (\Delta_g + a_2)\, u$ for all $u \in C^4(M)$ with $a_1, a_2 \in C^\infty(M)$ such that the two operators on $H^1(M)$ are coercive if and only if there exists $f \in C^\infty(M)$ such that

$$\begin{cases} A = f\ g, & \text{with } g \text{ the metric} \\ f^2 + 2\Delta_g f - 4a = K^2 \geq 0 & \text{a constant} \end{cases} \tag{II.77}$$

Proof. The "only if" part comes by developing the product. If $P_g = (\Delta_g + a_1) \circ (\Delta_g + a_2)$ then on one hand

$$\begin{aligned} P_g u &= (\Delta_g + a_1) \circ (\Delta_g u + a_2 u) = \Delta_g^2 u + a_2 \Delta_g u - 2\left(\nabla a_2, \nabla u\right)_g + \Delta_g a_2 u + a_1 \Delta_g u + a_1 a_2 u \\ &= \Delta_g^2 u + (a_1 + a_2)\Delta_g u - 2\left(\nabla a_2, \nabla u\right)_g + \left(\Delta_g a_2 + a_1 a_2\right) u, \end{aligned} \tag{II.78}$$

but on the other hand,

$$\begin{aligned} P_g u &= \Delta_g^2 u - \mathrm{div}_g\left(A(\nabla u)\right) + au \\ &= \Delta_g^2 u - (\nabla_i A^{ij})\nabla_j u - A^{ij}\nabla_i(\nabla u)_j + au. \end{aligned} \tag{II.79}$$

By identifying the terms, one gets $A = (a_1 + a_2)g$, since by definition $\Delta_g u = -g^{ij}\nabla_i(\nabla u)_j$. We can then define $f = -(a_1 + a_2) \in C^\infty(M)$, to get $A = f\ g$, but we also must have $2\nabla a_2 = g_{ij}\nabla_k A^{jk} dx^i$ and $a = \Delta_g a_2 + a_1 a_2$ so that

$$g_{ij}\nabla_k A^{jk} = g_{ij}\nabla_k\left((a_1 + a_2)g^{jk}\right) = \nabla_i a_1 + \nabla_i a_2 = 2\nabla_i a_2 \tag{II.80}$$

if and only if $\nabla a_1 = \nabla a_2$. That means there is $K \in \mathbb{R}$ such that $a_2 = a_1 + K$, and in that case $a = \Delta_g a_1 + a_1^2 + K a_1$. Computing the expression for f,

$$f^2 + 2\Delta_g f - 4a = (2a_1 + K)^2 + 4\Delta_g a_1 - 4\Delta_g a_1 - 4a_1^2 - 4K a_1 = K^2 \geq 0 \tag{II.81}$$

a constant as stated.

Conversely, there is $f \in C^\infty(M)$ such that $A = f\ g$ and $f^2 + 2\Delta_g f - 4a = K^2$ constant, then

$$\begin{aligned} P_g u &= \Delta_g^2 u - g(\nabla f, \nabla u) + f\Delta_g u + au \\ &= \Delta_g^2 u - g(\nabla f, \nabla u) + f\Delta_g u + \left(\frac{f^2 + 2\Delta_g f - K^2}{4}\right) u \\ &= \left(\Delta_g + \frac{f}{2} - \frac{K}{2}\right) \circ \left(\Delta_g + \frac{f}{2} + \frac{K}{2}\right) u \end{aligned} \tag{II.82}$$

□

Corollary 3.8.i Let $b, c \in \mathbb{R}$ and the operator $P_g = \Delta_g^2 + b\Delta_g + c$, then P_g can be written as the product of two operators if and only if $\frac{b^2}{4} - c \geq 0$.

Corollary 3.8.ii The Paneitz operator can be expressed as a product of two second-degree Laplacian operators if and only if the metric is Einstein.

Proof. We already saw that in the case the metric is Einstein, P_g^n the Paneitz operator in dimension n then has constant coefficients and

$$\left(\frac{n^2 - 2n - 4}{2}\frac{\lambda}{n-1}\right)^2 - 4\left(\frac{n(n-4)(n^2-4)}{16}\frac{\lambda^2}{(n-1)^2}\right) > 0 \tag{II.83}$$

for all $n \geq 4$. Reciprocally if P_g^n is the product of two operators, by Theorem 3.8 there is $f \in C^\infty(M)$ such that

$$\frac{(n-2)^2 + 4}{2(n-1)(n-2)} Sc_g g - \frac{4}{n-2} Ricc_g = f\ g. \tag{II.84}$$

This means we have $\tilde{f} \in C^\infty(M)$ such that $Ricc_g = \tilde{f} g$. □

Existence of a minimizer. The only thing left to discuss before ending the analysis of the resolution of general bi-Laplacian equations such as (II.50) is whether a minimizer exists. In the sub-critical case, the minimization of Rayleigh's quotient is always possible.

Proposition 3.9. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, let A be a symmetric bi-linear map on vectors of M and $a, f \in C^\infty(M)$ such that $f > 0$. If $2 \leq q < 2^\#$, then the associated Rayleigh quotient

$$R_{f,q}(u) = \frac{\int_M u P_g u dv_g}{\int_M f |u|^q dv_g}^{2/q} \tag{II.85}$$

attains its minimum, and defining $\mu_q(f) := \inf_{u \in H^2(M) \setminus \{0\}} R_{f,q}(u) \in \mathbb{R}$, there is $u \in H^2(M) \setminus \{0\}$ such that $R_{f,q}(u) = \mu_q(f)$.

We begin by showing this useful result.

Lemma 3.10. Let us suppose that $2 \leq q \leq 2^\#$, and that $R_{f,q}$ has a lower bound. Then if (u_i) is a minimizing sequence in $H^2(M) \setminus \{0\}$ for $R_{f,q}$, there is a $C \in \mathbb{R}$ such that $\|u_i\|_{H^2} \leq C$.

Proof. We start by showing the general fact that for all $u \in H^2(M)$, $\forall\ \varepsilon > 0$, there is $C > 0$ such that

$$\|\nabla u\|_{L^2} \leq \varepsilon \|\Delta_g u\|_{L^2} + C \|u\|_{L^2} . \tag{II.86}$$

By contradiction, if for all $k = 1, 2, \dots$ we have a $u_k \in H^2(M)$ such that

$$\|\nabla u_k\|_{L^2} > \varepsilon \|\Delta_g u_k\|_{L^2} + k \|u_k\|_{L^2} , \tag{II.87}$$

one can assume without loss of generality that $\|\nabla u_k\|_{L^2} = 1$. We get

$$\begin{aligned} &\|u_k\|_{L^2} \leq \frac{1}{k} \left(\varepsilon \|\Delta_g u_k\|_{L^2} + k \|u_k\|_{L^2} \right) \leq \frac{1}{k} \quad \forall\ k, \text{ and similarly} \\ &\|\Delta_g u_k\|_{L^2} \leq \frac{1}{\varepsilon} \end{aligned} \tag{II.88}$$

so that $\|u_k\|_{H^2} \leq \frac{1}{\varepsilon} + 1 + \frac{1}{k}$ for all k. In other words (u_k) is a bounded sequence of $H^2(M)$, we can assume that there is $u \in H^2(M)$ such that up to a sub-sequence $u_k \rightharpoonup u$ in H^2. On one hand since $H^2 \hookrightarrow H^1$ is compact, there exists a new subsequence that we rename u_k such that $u_k \to u$ strongly in $H^1(M)$. But then

$$\begin{cases} \|u\|_{L^2} = \lim_{k \to \infty} \|u_k\|_{L^2} = 0 \\ \|\nabla u\|_{L^2} = \lim_{k \to \infty} \|\nabla u_k\|_{L^2} = 1 \end{cases} \tag{II.89}$$

and that gives us a contradiction.

So (II.86) is true, we then always have

$$\begin{aligned} R_{f,q}(u) &= \frac{\int_M \left((\Delta_g u)^2 + A(\nabla u, \nabla u) + au^2\right) dv_g}{\left(\int_M f\,|u|^q\, dv_g\right)^{2/q}} \\ &\geq \frac{\int_M (\Delta_g u)^2 dv_g - \|A\|_\infty \|\nabla u\|_{L^2}^2 - \|a\|_\infty \|u\|_{L^2}^2}{\left(\int_M f\,|u|^q\, dv_g\right)^{2/q}} \\ &\geq \frac{(1-C\varepsilon)\,\|\Delta_g u\|_{L^2}}{\left(\int_M f\,|u|^q\, dv_g\right)^{2/q}} - \frac{(C' + \|a\|_\infty)\,\|u\|_{L^2}^2}{(\inf_M f)^{2/q}\,\|u\|_{L^q}^2} \\ &\geq \frac{(1-C\varepsilon)\,\|\Delta_g u\|_{L^2}}{\left(\int_M f\,|u|^q\, dv_g\right)^{2/q}} - \frac{C'\mathrm{Vol}_g(M)^{2-\frac{4}{q}}}{(\inf_M f)^{2/q}}. \end{aligned} \tag{II.90}$$

Now if (u_i) is a minimizing sequence for $R_{f,q}$, we write $\mu_q(f) := \inf_{u\in H^2(M)\setminus\{0\}} R_{f,q}u$, we have $\lim_{i\to\infty} R_{f,q}(u_i) = \mu_q(f)$. Without loss of generality one can assume $\int_M f\,|u|^q\, dv_g = 1$. Then

$$\|u_i\|_{L^2} \leq \mathrm{Vol}_g(M)^{\frac{1}{2}-\frac{1}{q}}\,\|u_i\|_{L^q} \leq \frac{\mathrm{Vol}_g(M)^{\frac{1}{2}-\frac{1}{q}}}{(\inf_M f)^{1/q}} \tag{II.91}$$

and taking $\varepsilon > 0$ small enough such that $C\varepsilon > 0$, we get from (II.90)

$$\|\Delta_g u_i\|_{L^2}^2 \leq C, \tag{II.92}$$

since $R_{f,q}(u_i) = \mu_q(f) + o(1)$. Putting this together, and with (II.86), we have

$$\|u_i\|_{H^2} \leq C \quad \text{for all } i. \tag{II.93}$$

□

Proof (of Proposition 3.9). The proof follows a standard variational method. First we show that $R_{f,q}$ has a lower bound, coming back to (II.90) in the preceding proof, so that we can take (u_i) a minimizing sequence for $R_{f,q}$ in $H^2(M)\setminus\{0\}$, such that $\lim_{i\to\infty} R_{f,q}(u_i) = \mu_q(f)$. Then by the Lemma (u_i) is uniformly bounded in $H^2(M)$, we can assume it converge weakly to $u \in H^2(M)$. But then since $H^2(M) \hookrightarrow L^q(M)$ is compact for $q < 2^\#$, we have $u_i \to u$ strongly in $L^q(M)$, and

$$\int_M f\,|u|^q\, dv_g = \lim_{i\to\infty} \int_M f\,|u_i|^q\, dv_g = 1, \tag{II.94}$$

so that $u \not\equiv 0$. We also know that the embedding $H^2(M) \hookrightarrow H^1(M)$ is compact, and we can assume $u_i \to u$ in $H^1(M)$. With that we define $\theta_i := u_i - u \in H^2(M)$, $u_i = \theta_i + u$ and we compute

$$\begin{aligned} \int_M u_i P_g u_i dv_g = \int_M uP_g u dv_g + 2\int_M (\Delta_g u \Delta_g \theta_i + A(\nabla u, \nabla\theta_i) + au\theta_i)\, dv_g \\ + \int_M \left((\Delta_g\theta_i)^2 + A(\nabla\theta_i, \nabla\theta_i) + a\theta_i^2\right) dv_g. \end{aligned} \tag{II.95}$$

Using that $\theta_i \to 0$ in $H^1(M)$ and that $\theta_i \rightharpoonup 0$ in $H^2(M)$, we have

$$\int_M u_i P_g u_i dv_g = \int_M uP_g u dv_g + \int_M (\Delta_g\theta_i)^2 dv_g + o(1), \tag{II.96}$$

so that

$$\mu_q(f) = \int_M uP_g u dv_g + \int_M (\Delta_g \theta_i)^2 dv_g + o(1). \tag{II.97}$$

But since $u \not\equiv 0$, by the definition of $\mu_q(f)$ we know $R_{f,q}(u) \geq \mu_q(f)$, and since $\int_M f\,|u|^q\,dv_g = 1$,

$$\begin{aligned} &R_{f,q}(u) = \int_M uP_g u \geq \mu_q(f) \\ \Rightarrow\ &\int_M uP_g u dv_g + \int_M (\Delta_g \theta_i)^2 dv_g + o(1) \leq \int_M uP_g u dv_g, \end{aligned} \tag{II.98}$$

we must have $\int_M (\Delta_g \theta_i)^2 dv_g \to 0$, and thus $\mu_q(f) = R_{f,q}(u)$. We found a $u \in H^2(M) \setminus \{0\}$ that reaches the minimum of $R_{f,q}$. □

In the critical case proving the existence of a minimizer is much more complicated, since one cannot prove that the weak limit of a minimizing sequence is non-zero. Even having a bounded minimizing sequence for $R_f = R_{f,2^\#}$, would only give us a weak limit without necessarily being able to conclude further. This comes from the lack of compactness of the embedding $H^2(M) \hookrightarrow L^{2^\#}(M)$. Actually, even with reasonable hypothesis on the operator P_g, such as coercivity and the pointwise comparison principle, a minimizer $u \in H^2(M)$ of R_f may not exist.

We will see that the Rayleigh's quotient is closely connected to the best constant for the embedding $H^2(M) \hookrightarrow L^{2^\#}(M)$. There exists indeed $C > 0$ such that

$$\|u\|^2_{L^{2^\#}} \leq C \|u\|^2_{H^2} \leq C \left(\|\Delta_g u\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2} \right), \tag{II.99}$$

and we are particularly interested in optimizing the constant in front of $\|\Delta_g u\|_{L^2}$.

Definition 3.11. Let us define the optimal constant

$$\begin{aligned} K_n := \inf\{C \in \mathbb{R} \quad &\text{s.t. } \exists\, C_0 \in \mathbb{R} \text{ with} \\ &\|u\|^2_{L^{2^\#}} \leq C \|\Delta_g u\|^2_{L^2} + C_0 \left(\|\nabla u\|_{L^2} + \|u\|^2_{L^2} \right) \quad \forall\ u \in H^2(M)\} \end{aligned}$$

Hebey (2003) has shown that the infimum is reached, that is there exists $C_0 > 0$ such that

$$\left(\int_M u^{2^\#} dv_g \right)^{2/2^\#} \leq K_n \int_M (\Delta_g u)^2 dv_g + C_0 \int_M \left(|\nabla u|^2 + u^2 \right) dv_g \quad \forall\ u \in H^2(M). \tag{II.100}$$

It turns out that both $\|\Delta_g u\|_{L^2}$ and $\|u\|_{L^{2^\#}}$ are almost invariant under the same scaling operation.

Definition 3.12. Let $x \in M$, a *weight* $\mu > 0$ and a smooth function $u : M \to \mathbb{R}$, we define

$$R^\mu_x u(y) := \mu^{\frac{n-4}{2}} u(\exp_x(\mu y)) \tag{II.101}$$

for all $y \in \mathbb{R}^n$ such that $\mu\,|y| < i_g$.

This corresponds to zooming on a point $x \in M$ to end up in the local chart around it, then renormalizing the function. With this we can compute

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(-\Delta R_x^\mu u\right)^2 dy^n = \int_M (\Delta_g u)^2 dv_g + o(1) \\
&\int_{\mathbb{R}^n} \left(R_x^\mu u\right)^{2^\#} dy^n = \int_M u^{2^\#} dv_g + o(1), \qquad \text{but} \\
&\int_{\mathbb{R}^n} \left(R_x^\mu u\right)^2 dy^n = \mu^{-4} \int_M u^2 dv_g + o(1) \\
&\int_{\mathbb{R}^n} \left|\nabla R_x^\mu u\right|^2 dy^n = \mu^{-2} \int_M |\nabla u|^2 dv_g + o(1)
\end{aligned} \tag{II.102}$$

where integrals in $\mathbb{R}^n$ are actually taken on the bounded range set of the considered function, and $o(1) \to 0$ as $\mu \to 0$ are residual terms coming from the curvature of the manifold. Then

$$\begin{aligned}
\|u\|_{H^2(M)} &= \|-\Delta R_x^\mu u\|_{L^2(\mathbb{R}^n)} + \mu \|\nabla R_x^\mu u\|_{L^2(\mathbb{R}^n)} + \mu^2 \|R_x^\mu u\|_{L^2(\mathbb{R}^n)} \\
&= \|-\Delta R_x^\mu u\|_{L^2(\mathbb{R}^n)} + o(1)
\end{aligned} \tag{II.103}$$

for $\mu \to 0$.

If u_0 reaches the optimal constant, that is

$$\begin{aligned}
&\|u_0\|_{L^{2^\#}}^2 = K_n \|\Delta_g u_0\|_{L^2}^2 + C_0 \left(\|\nabla u_0\|_{L^2}^2 + \|u_0\|_{L^2}^2\right) \\
\Rightarrow\ &\|R_x^\mu u\|_{L^{2^\#}(\mathbb{R}^n)}^2 = K_n \|-\Delta R_x^\mu u\|_{L^2(\mathbb{R}^n)}^2 + C_0 \left(\mu^2 \|\nabla R_x^\mu u\|_{L^2(\mathbb{R}^n)}^2 + \mu^4 \|R_x^\mu u\|_{L^2(\mathbb{R}^n)}^2\right) + o(1)
\end{aligned} \tag{II.104}$$

by taking $\mu \to 0$, $R_x^\mu u_0$ will be defined on whole $\mathbb{R}^n$ and asymptotically

$$\|R_x^\mu u\|_{L^{2^\#}(\mathbb{R}^n)}^2 = K_n \|-\Delta R_x^\mu u\|_{L^2(\mathbb{R}^n)}^2 + o(1) \tag{II.105}$$

with $o(1) \to 0$ as $\mu \to 0$.

But we know that on $\mathbb{R}^n$ $\|u\|_{L^{2^\#}}^2 \leq K_n \|-\Delta u\|_{L^2}^2$ for all $u \in C_0^\infty(\mathbb{R}^n)$, the Equation (II.103) leads us to define a new function space, $D^2(\mathbb{R}^n)$ the completion of $C_0^\infty(\mathbb{R}^n)$ for the norm

$$\|u\|_{D^2(\mathbb{R}^n)} := \|-\Delta u\|_{L^2(\mathbb{R}^n)}. \tag{II.106}$$

In the literature, this is referred to as a *Beppo-Levi* space, and is sometimes written $\dot{H}^{k,p}$ in the general case. Formally, the rescaling of $u \in H^2(M)$ leads in the limit to $R_x^\mu u \in D^2(\mathbb{R}^n)$. And if $u_0 \in D^2(\mathbb{R}^n)$ reaches the optimal constant,

$$\|u_0\|_{L^{2^\#}}^2 = K_n \|-\Delta u_0\|_{L^2}^2, \tag{II.107}$$

then u_0 minimizes Rayleigh's quotient

$$\tilde{R}_1(u) := \frac{\int_{\mathbb{R}^n} (-\Delta u)^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2^\#} dx\right)^{2/2^\#}} \tag{II.108}$$

associated to the operator $(-\Delta)^2$ on $\mathbb{R}^n$. We then have

$$\inf \tilde{R}_1(u) = \tilde{R}_1(u_0) = \frac{1}{K_n} \tag{II.109}$$

and u_0 leads to a solution to the limiting critical equation

$$(-\Delta)^2 u = u^{2^\# - 1}. \tag{II.110}$$

By Lin's classification (Lin, 1998) of solutions, we know that the functions satisfying this equation, and thus realizing the infimum of $\tilde{R}_1$ are essentially of the shape of the ground state

$$\mathrm{b}(x) = \left(1 + \frac{|x|^2}{\sqrt{\lambda_n}}\right)^{-\frac{n-4}{2}}, \tag{II.111}$$

with $\lambda_n = n(n-4)(n^2-4)$. This function exactly looks, up to a fixed scaling, like the conformal term obtained when transforming the flat metric through stereographic projection in (II.22). We can then translate and concentrate this function with a weight $\mu > 0$ thanks to the symmetry by the scaling operation 3.12 of the limiting critical equation,

$$\mathrm{b}_{x_0,\mu}(x) = \left(\frac{\mu}{\mu^2 + \frac{|x-x_0|^2}{\sqrt{\lambda_n}}}\right)^{\frac{n-4}{2}} = \mu^{-\frac{n-4}{2}} \mathrm{b}\left(\frac{x - x_0}{\mu}\right) \in C^\infty(\mathbb{R}^n). \tag{II.112}$$

This function has the property to be solution to (II.110), and to reach the optimal Sobolev constant,

$$\left(\int_{\mathbb{R}^n} \mathrm{b}_{x_0,\mu}^{2^\#}\right)^{\frac{2}{2^\#}} = K_n \left(\int_{\mathbb{R}^n} (-\Delta \mathrm{b}_{x_0,\mu})^2\right) = K_n \int_{\mathbb{R}^n} \mathrm{b}_{x_0,\mu}^{2^\#}. \tag{II.113}$$

The latter allows to calculate $\int_{\mathbb{R}^n} \mathrm{b}_{x_0,\mu}^{2^\#} = K_n^{-\frac{n}{4}}$.

The following existence result allows us to make the final connection between the optimal constant K_n and our critical problem.

Theorem 3.13. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, let A be a symmetric bi-linear map on vectors of M and $a, f \in C^\infty(M)$ such that $f > 0$. If $P_g = \Delta_g^2 - \mathrm{div}_g(\nabla \cdot) + a$ is coercive, then if

$$\inf_{u \in H^2(M)\setminus\{0\}} R_f(u) < \frac{1}{\left(\sup_M f\right)^{2/2^\#} K_n} \tag{II.114}$$

there exists $u \in C^4(M)$, with $u \not\equiv 0$ and $P_g u = f\,|u|^{2^\#-2}\,u$. Moreover, the infimum is reached.

Since P_g is assumed to be coercive, we already saw that

$$\mu = \inf_{u \in H^2(M)\setminus\{0\}} R_f(u) \geq \frac{\lambda}{\|f\|_\infty^{2/2^\#}}, \tag{II.115}$$

so that $\mu > 0$. We then take a minimizing sequence (u_i) of functions in $H^2(M)\setminus\{0\}$, with $R_f(u_i) \to \mu$. We can assume, up to multiplying by a constant, that $\int_M f\,|u_i|^{2^\#}\,dv_g = 1$ for all i. By Lemma

3.10, we know the u_i are bounded, there is $u \in H^2(M)$ such that $u_i \rightharpoonup u$ in $H^2(M)$, and $u_i \to u$ strongly in $H^1(M)$ still up to a new subsequence. Following the proof of Proposition 3.9, we define $\theta_i = u_i - u \in H^2(M)$, and we have

$$\mu = \mu_{2^\#}(f) = \int_M uP_g u dv_g + \int_M (\Delta_g \theta_i)^2 dv_g + o(1). \tag{II.116}$$

Here however we do not have the compactness for the embedding $H^2(M) \hookrightarrow L^{2^\#}(M)$ anymore, so we cannot use the same argument as in 3.9. We then refer to Robert (2009, Theorem 5.2) for a complete proof of the proposition.

Corollary 3.13.i Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, and $b, c \in \mathbb{R}$ with $b, c > 0$ such that $\frac{b^2}{4} - c \geq 0$. Let also $f \in C^\infty(M)$ be such that $f > 0$ and the operator $P_g = \Delta_g^2 + b\Delta_g + c$. If

$$\inf_{u \in H^2(M)\setminus\{0\}} \frac{\int_M \left((\Delta_g u)^2 + b\left|\nabla u\right|^2 + cu^2\right) dv_g}{\left(\int_M f \left|u\right|^{2^\#} dv_g\right)^{2^\#/2}} < \frac{1}{\left(\sup_M f\right)^{2^\#/2} K_n} \tag{II.117}$$

then there is a $u \in C^4(M)$ such that $u > 0$ and $P_g u = f u^{2^\#-1}$.

Proof. We start by noticing that for all $u \in H^2(M)$,

$$\int_M \left((\Delta_g u)^2 + b\left|\nabla u\right|^2 + cu^2\right) dv_g \geq a \left\|u\right\|_{L^2} \tag{II.118}$$

so that P_g is coercive. By Theorem 3.13 just above there is a $u \in C^4(M)$ such that $P_g u = f u \left|u\right|^{2^\#-2} u$. But now using Corollary 3.8.i we know that P_g is the product of two coercive operators, and thus it satisfies the pointwise comparison principle. Theorem 3.6 tells us that either $u > 0$ or $u < 0$, up to multiplying it by (-1), we get $u > 0$ in M with $P_g u = f u^{2^\#-1}$. □

We can then re-write condition (II.117) to get more precise conditions on our choice of function (see also Esposito & Robert, 2002).

Corollary 3.13.ii Let $b, c > 0$ be real numbers such that $\frac{b^2}{4} - c \geq 0$ and $f \in C^\infty(M)$ such that $f > 0$. Writing $x_0 \in M$ such that $f(x_0) = \sup_M f(x)$, if

$$8n(n-1)b + (n-6)(n+2)(n-4)\frac{\Delta_g f(x_0)}{f(x_0)} - 4(n^2 - 2n - 4)Sc_g(x_0) < 0 \tag{II.119}$$

then there exists $u \in C^4(M)$ such that $u > 0$ and $\Delta_g^2 u + b\Delta_g u + cu = f u^{2^\#-1}$.

4 Stability of the equations

We just saw that when R_f remains below some threshold, we obtain a minimizer for that functional. Another classical way to study the existence of solutions to equations of the type

$$P_g u = \Delta_g^2 u - \mathrm{div}_g\left(A(\nabla u)\right) + au = u^{2^\#-1} \qquad \text{with } u > 0 \text{ in M}, \tag{II.120}$$

is to consider the associated variational problem. We introduce the functional

$$J(u) = \frac{1}{2}\int_M uP_g u dv_g - \frac{1}{2^\#}\int_M |u|^{2^\#} dv_g \qquad \forall\ u \in H^2(M), \tag{II.121}$$

one can see that J is differentiable and that

$$\langle DJ(u), \varphi\rangle = \int_M (\Delta_g u \Delta_g \varphi + A(\nabla u, \nabla\varphi) + au\varphi)\, dv_g - \int_M |u|^{2^\#-2} u\varphi\ dv_g \quad \forall\ u, \varphi \in H^2(M). \tag{II.122}$$

In other words, the weak formulation of equation (II.120) is the Euler-Lagrange equation searching for critical points of the functional $J(u)$. For this reason, J is sometimes referred to as the *energy* functional. A fundamental feature of the latter is precisely the existence of sequences of functions that are asymptotically critical but which will not converge in H^2.

In this section, still coming mainly from Robert (2009), we will start by introducing the concept of a Palais-Smale sequence for the problem, that will finally allow us to reach an actual weak solution, but with a particularity discussed in the set-up of the Theorem 4.6. In a second time, we will present the main matter for the following chapters, that is focusing on sequences of solutions to equations approaching the one of interest, and seeing how the lack of compactness of the problem translate in the stability of the solutions.

Concentration bubbles. The functions defined in (II.112) can similarly be defined in a Riemannian setup.

Definition 4.1. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function around $x_0 \in M$ and $\mu > 0$. A *bubble* is defined to be the function

$$B_{x_0,\mu}(x) = \eta_{x_0}(x)\left(\frac{\mu}{\mu^2 + d_g(x_0,x)^2/\sqrt{\lambda_n}}\right)^{\frac{n-4}{2}}$$

where $\eta_{x_0}(x) := \eta(d_g(x_0,x))$ and $\lambda_n := n(n-4)(n^2-4)$.

For a sequence of converging points in M, (x_α), and (μ_α) such that $\mu_\alpha \to 0$, we see that the bubble B_{x_α,μ_α} becomes essentially zero everywhere except at $x_0 = \lim_{\alpha\to\infty} x_\alpha$ where it blows up. Such a sequence of *bubbles* concentrates inside balls of ever shrinking radius. The invariance of the norms through rescaling shows that they converge to 0 weakly, but not strongly as $\alpha \to \infty$.

Remark : The fact that B_{x_α,μ_α} converges pointwise to zero except at x_0 explains why the choice of a particular cutoff function η does not matter. Around the center, B_{x_α,μ_α} behaves as the Euclidean bubble (II.111), solution to the limiting critical equation on $\mathbb{R}^n$.

Unlike in the Euclidean setting, here these Riemannian bubbles are only solutions to the limiting critical equation on M at first order. The presence of the geometry adds non-trivial terms :

$$\Delta_g^2(B_{x_\alpha,\mu_\alpha}) = (B_{x_\alpha,\mu_\alpha})^{2^\#-1} + o(1). \tag{II.123}$$

Let us be more specific with some main properties of these functions.

Proposition 4.2. Let (M,g) be a Riemannian manifold of dimension $n \geq 5$, and $(x_\alpha)_{\alpha\in\mathbb{N}}$ a converging sequence of points of M, let $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ a sequence of real positive numbers such that $\mu_\alpha \to 0$. Then $(B_{x_\alpha,\mu_\alpha})_\alpha$ is a sequence of $C^\infty(M)$ functions such that

i. they are aysmptotically critical for the functional, $\lim_{\alpha\to\infty} DJ(B_{x_\alpha,\mu_\alpha}) = 0$ in $H^2(M)'$;

ii. they weakly tend to 0, $B_{x_\alpha,\mu_\alpha} \rightharpoonup 0$ in $H^2(M)$;

iii. $\lim_{\alpha\to\infty} J(B_{x_\alpha,\mu_\alpha}) = \frac{2}{n}K_n^{-\frac{n}{4}}$, so that we know it cannot converge strongly to 0 in $H^2(M)$.

Proof. We will write in the following $B_\alpha := B_{x_\alpha,\mu_\alpha}$ to make the notations easier, and we will write δ the radius such that $\eta \equiv 1$ if $|x| < \delta/2$ and $\eta \equiv 0$ when $|x| > \delta$, assuming that $\delta < i_g(M)$ the injectivity radius. We start by noticing that

$$\begin{aligned}\int_M |B_\alpha|^2\, dv_g &\leq \int_{B_{x_\alpha}(\delta)} \left(\frac{\mu_\alpha}{\mu_\alpha^2 + d_g(x_\alpha,x)^2/\sqrt{\lambda_n}}\right)^{n-4} dv_g \\ &\leq \int_{B_0(\delta)} \left(\frac{\mu_\alpha}{\mu_\alpha^2 + |x|^2/\sqrt{\lambda_n}}\right)^{n-4} \sqrt{|g_\alpha(x)|}\, dx\end{aligned} \tag{II.124}$$

on the local exponential chart, with $g_\alpha := \exp_{x_\alpha}^* g$ defined on $B_0(\delta)$. By identification of $T_{x_\alpha}M$ with $\mathbb{R}^n$, we have $(g_\alpha)_{ij}(0) = \delta_{ij}$ and $\nabla_i = \frac{\partial}{\partial x^i}$ the Levi-Civita connection. Then

$$\int_M |B_\alpha|^2\, dv_g \leq \mu_\alpha^4 \int_{B_0(\delta/\mu_\alpha)} \left(1 + \frac{|x|^2}{\sqrt{\lambda_n}}\right)^{-(n-4)} \sqrt{|g_\alpha(\mu_\alpha x)|}\, dx = o(1) \tag{II.125}$$

as $\alpha \to \infty$. In the same way

$$\begin{aligned}\int_M |\nabla B_\alpha|^2\, dv_g &= \int_{B_{x_\alpha}(\delta)} |\nabla B_\alpha|^2\, dv_g + \int_{M\setminus B_{x_\alpha}(\delta)} |\nabla B_\alpha|^2\, dv_g \\ &\leq \mu_\alpha^2 \int_{B_0(\delta/\mu_a)} (g_\alpha(\mu_\alpha x))(\nabla \mathrm{b}, \nabla \mathrm{b})\sqrt{|g_\alpha(\mu_\alpha x)|}\, dx + C\mu_\alpha^{n-4} = o(1)\end{aligned} \tag{II.126}$$

using the explicit formulation of b.

One then gets

$$\begin{aligned}\langle DJ(B_\alpha), \varphi\rangle &= \int_M (\Delta_g B_\alpha \Delta_g \varphi + A(\nabla B_\alpha, \nabla\varphi) + aB_\alpha\varphi) dv_g - \int_M B_\alpha^{2^\#-1}\varphi dv_g \\ &= \int_M \Delta_g B_\alpha \Delta_g \varphi\, dv_g - \int_M B_\alpha^{2^\#-1}\varphi\, dv_g + o(1)\,\|\varphi\|_{H^2}\end{aligned} \tag{II.127}$$

for all $\varphi \in C^\infty(M)$, where $o(1) \xrightarrow{\alpha\to\infty} 0$ independently of φ. The goal now is to be able to express the integrals on M remaining in (II.127) involving $B_\alpha = B_{x_\alpha,\mu_\alpha}$ as integrals on $\mathbb{R}^n$ of b.

To do so we have to split the integral in two parts, inside and outside a ball of radius $R\mu_\alpha$. Let us fix $R > 0$, we have on one hand

$$\begin{aligned}\int_{M\setminus B_{x_\alpha}(R\mu_\alpha)} B_\alpha^{2^\#} dv_g &= \int_{B_{x_\alpha}(\delta/2)\setminus B_{x_\alpha}(R\mu_\alpha)} \left(\frac{\mu_\alpha}{\mu_\alpha^2 + d_g(x_\alpha,x)^2/\sqrt{\lambda_n}}\right)^n dv_g(x) \\ &\quad + \int_{M\setminus B_{x_\alpha}(\delta/2)} (\eta_{x_\alpha}(x))^{2^\#} \left(\frac{\mu_\alpha}{\mu_\alpha^2 + d_g(x_\alpha,x)^2/\sqrt{\lambda_n}}\right)^n dv_g(x).\end{aligned} \tag{II.128}$$

On the local exponential map, by changing the scale we have

$$\int_{M\setminus B_{x_\alpha}(R\mu_\alpha)} B_\alpha^{2^\#} dv_g = \int_{B_0(\delta/2\ \mu_\alpha^{-1}))\setminus B_0(R)} \left(1+\frac{|x|^2}{\sqrt{\lambda_n}}\right)^{-n} \sqrt{|g_\alpha(\mu_\alpha x)|}\ dx + O(\mu_\alpha^n), \tag{II.129}$$

but $\mu_\alpha \to 0$ and writing ξ the Euclidean metric, $g_\alpha \to \xi$ in $C^2_{loc}(\mathbb{R}^n)$, so that

$$\int_{M\setminus B_{x_\alpha}(R\mu_\alpha)} B_\alpha^{2^\#} dv_g = \int_{\mathbb{R}^n\setminus B_0(R)} \mathrm{b}^{2^\#} dx + o(1). \tag{II.130}$$

On the other hand on $B_{x_\alpha}(R\mu_\alpha)$, for any $\psi \in C^\infty(M)$,

$$\begin{aligned}\int_{B_{x_\alpha}(R\mu_\alpha)} B_\alpha^{2^\#-1}\psi dv_g &= \int_{B_0(R\mu_\alpha)} B_\alpha(\exp_{x_\alpha}(x))^{2^\#-1}\psi(\exp_{x_\alpha}(x))\sqrt{|g_\alpha(x)|}dx \\ &= \int_{B_0(R\mu_\alpha)} B_\alpha(\exp_{x_\alpha}(x))^{2^\#-1}\psi(\exp_{x_\alpha}(x))dx + o(1)\int_{B_{x_\alpha}(R\mu_\alpha)} B_\alpha^{2^\#-1}\psi dv_g \\ &= \int_{B_0(R)} \mu^{\frac{n-4}{2}}\mathrm{b}(x)^{2^\#-1}\psi(\exp_{x_\alpha}(\mu_\alpha x)dx + o(1)\left\|B_\alpha\right\|_{L^{2^\#}}^{2^\#-1}\left\|\psi\right\|_{L^{2^\#}}\end{aligned} \tag{II.131}$$

using Hölder in the last inequality.

Regarding $\Delta_g B_\alpha$, we decompose the integral in the same way, by imitating what we got in (II.128,II.129)

$$\begin{aligned}\int_{M\setminus B_{x_\alpha}(R\mu_\alpha)} (\Delta_g B_\alpha)^2 dv_g &= \int_{B_0(\delta/2\ \mu_\alpha^{-1})\setminus B_0(R)} \left(\mathrm{div}_{g_\alpha}\left(\nabla_{g_\alpha}\mathrm{b}\right)(\mu_\alpha x)\right)^2 \sqrt{|g_\alpha(\mu_\alpha x)|}\ dx + O(\mu_\alpha^{n-4}) \\ &= \int_{\mathbb{R}^n\setminus B_0(R)} (\Delta_\xi \mathrm{b})^2 dx + o(1)\end{aligned} \tag{II.132}$$

as $\alpha \to \infty$. We also have on $B_{x_\alpha}(R\mu_\alpha)$, for all $\psi \in C^\infty(M)$,

$$\int_{B_{x_\alpha}(R\mu_\alpha)} \Delta_g B_\alpha \Delta_g \psi dv_g = \int_{B_0(R)} \Delta_\xi \mathrm{b}\Delta_\xi\left(\mu_\alpha^{\frac{n-4}{2}}\psi\circ\exp_{x_\alpha}(\mu_\alpha x)\right)dx + o(1)\left\|B_\alpha\right\|_{H^2}\left\|\psi\right\|_{H^2}, \tag{II.133}$$

that we will show hereafter.

Using (II.131) and (II.133) and that B_α is a $L^{2^\#}$ function in (II.130) and (II.132), we get

$$\begin{aligned}\langle DJ(B_\alpha), \varphi\rangle &= \int_M \Delta_g B_\alpha \Delta_g \varphi\ dv_g - \int_M B_\alpha^{2^\#-1}\varphi\ dv_g + o(1)\left\|\varphi\right\|_{H^2} \\ &= \int_{B_0(R)} \Delta_\xi \mathrm{b}\Delta_\xi\varphi_\alpha dx - \int_{B_0(R)} \mathrm{b}^{2^\#-1}\varphi_\alpha dx + \varepsilon_\alpha(R)\left\|\varphi\right\|_{H^2} + o(1)\left\|\varphi\right\|_{H^2}\end{aligned} \tag{II.134}$$

where $\varepsilon_\alpha(R)$ represents the integrals on $M\setminus B_{x_\alpha}(R\mu_\alpha)$ after using Hölder's inequality, with $\varepsilon_\alpha(R)\to 0$ as $R\to\infty$, and writing $\varphi_\alpha(x) := \mu_\alpha^{\frac{n-4}{2}}\varphi\circ\exp_{x_\alpha}(\mu_\alpha x)$ for all $x\in\mathbb{R}^n$ such that α is large enough

so that $|\mu_\alpha x| < i_g(M)$. Now similarly for b,

$$\left|\int_{\mathbb{R}^n\setminus B_0(R)} \Delta_\xi \mathrm{b} \Delta_\xi \varphi_\alpha dx\right| \leq \varepsilon(R) \|\varphi_\alpha\|_{D^2(\mathbb{R}^n)} \tag{II.135}$$

$$\left|\int_{\mathbb{R}^n\setminus B_0(R)} \mathrm{b}^{2^\#-1} \varphi_\alpha dx\right| \leq C\varepsilon(R) \|\varphi_\alpha\|_{D^2(\mathbb{R}^n)} \tag{II.136}$$

with $\varepsilon(R) \xrightarrow{R\to\infty} 0$, using Hölder's inequality, the fact that both b and $\Delta_\xi \mathrm{b}$ are $L^{2^\#}(\mathbb{R}^n)$ functions and the Sobolev's embedding inequality $D^2 \hookrightarrow L^{2^\#}$. Finally, noting that $\|\varphi_\alpha\|_{D^2(\mathbb{R}^n)} \leq C \|\varphi\|_{H^2}$, we have

$$\langle DJ(B_\alpha), \varphi\rangle = \int_{\mathbb{R}^n} \Delta_\xi \mathrm{b} \Delta_\xi \varphi_\alpha dx - \int_{\mathbb{R}^n} \mathrm{b}^{2^\#-1} \varphi_\alpha dx + \varepsilon_\alpha(R) \|\varphi\|_{H^2} + o(1) \|\varphi\|_{H^2}. \tag{II.137}$$

Since φ_α is still in $C_0^\infty(\mathbb{R}^n)$ for all α, we can use integration by part and the fact that b solves the limiting critical equation (II.110),

$$\int_{\mathbb{R}^n} \Delta \mathrm{b}\ \Delta\varphi_\alpha dx = \int_{\mathbb{R}^n} (-\Delta)^2 \mathrm{b}\ \varphi_\alpha dx = \int_{\mathbb{R}^n} \mathrm{b}^{2^\#-1} \varphi_\alpha dx. \tag{II.138}$$

Letting $R \to \infty$, we showed that $|\langle DJ(B_\alpha), \varphi\rangle| \leq o(1) \|\varphi\|_{H^2}$, so that $DJ(B_\alpha) \to 0$ in $H^2(M)'$ as $\alpha \to \infty$.

We also get for all $\psi \in C^\infty(M)$,

$$\begin{aligned}\left|\int_M \Delta_g B_\alpha \Delta_g \psi dv_g\right| &\leq \left|\int_{B_{x_\alpha}(R\mu_\alpha)} \Delta_g B_\alpha \Delta_g \psi dv_g\right| + \left|\int_{M\setminus B_{x_\alpha}(R\mu_\alpha)} \Delta_g B_\alpha \Delta_g \psi dv_g\right| \\ &\leq \|\Delta_g B_\alpha\|_{L^2(M)} \|\Delta_g \psi\|_{L^2(B_{x_\alpha}(R\mu_\alpha))} + \|\Delta_g B_\alpha\|_{L^2(M\setminus B_{x_\alpha}(R\mu_\alpha))} \|\Delta_g \psi\|_{L^2(M)} \\ &\leq C \|\Delta_g \psi\|_{L^2(B_{x_\alpha}(R\mu_\alpha))} + C \|\Delta_g B_\alpha\|_{L^2(M\setminus B_{x_\alpha}(R\mu_\alpha))} \qquad \forall\ R > 0.\end{aligned} \tag{II.139}$$

We then use that $\psi \in C^\infty(M)$, $\mu_\alpha \to 0$, and that by (II.132),

$$\lim_{R\to\infty} \lim_{\alpha\to\infty} \|\Delta_g B_\alpha\|_{L^2(M\setminus B_{x_\alpha}(R\mu_\alpha)} = 0 \tag{II.140}$$

using the expression of b, and we obtain

$$\int_M \Delta_g B_\alpha \Delta_g \psi dv_g \to 0 \qquad \forall\ \psi \in C^\infty(M) \tag{II.141}$$

letting $\alpha \to \infty$ and then $R \to \infty$. This, with the fact that we saw in (II.125,II.126) that $\|B_\alpha\|_{L^2}, \|\nabla B_\alpha\|_{L^2} \to 0$, tells that $B_\alpha \rightharpoonup 0$ in $H^2(M)$. Here we used the density of $C^\infty(M)$ in $H^2(M)$. That gives the second claim.

To conclude the proof, taking $\psi = B_\alpha$ in (II.131) and (II.133)

$$\begin{aligned}\int_M (\Delta_g B_\alpha)^2 dv_g &= \int_{B_0(R)} (\Delta_\xi \mathrm{b})^2 dx + \int_{\mathbb{R}^n\setminus B_0(R)} (\Delta_\xi \mathrm{b})^2 dx + o(1) \|B_\alpha\|_{H^2}^2 \\ &\xrightarrow{\alpha\to\infty} \int_{\mathbb{R}^n} (-\Delta \mathrm{b})^2 dx\end{aligned} \tag{II.142}$$

and $\displaystyle\int_M B_\alpha^{2^\#} dv_g \xrightarrow{\alpha\to\infty} \int_{\mathbb{R}^n} \mathrm{b}^{2^\#} dx.$

But

$$\begin{aligned} J(B_\alpha) &= \frac{1}{2}\int_M ((\Delta_g B_\alpha)^2 + A(\nabla B_\alpha, \nabla B_\alpha) + aB_\alpha^2)dv_g - \frac{1}{2^\#}\int_M |B_\alpha|^{2^\#} dv_g \\ &= \frac{1}{2}\int_{\mathbb{R}^n} (-\Delta \mathrm{b})^2 dx - \frac{1}{2^\#}\int_{\mathbb{R}^n} \mathrm{b}^{2^\#} dx + o(1). \end{aligned} \tag{II.143}$$

Since b is a solution to (II.110), testing the weak formulation of (II.110) against $\mathrm{b} \in C^\infty(M)$ itself, we obtain

$$\int_{\mathbb{R}^n} (-\Delta \mathrm{b})^2 dx = \int_{\mathbb{R}^n} \mathrm{b}^{2^\#} dx, \tag{II.144}$$

and since b attains the best Sobolev constant for the embedding of $D^2(\mathbb{R}^n)$ in $L^{2^\#}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \mathrm{b}^{2^\#} dx = K_n^{-\frac{n}{4}}$. Finally that gives

$$J(B_\alpha) \xrightarrow{\alpha\to\infty} \frac{2}{n} K_n^{-\frac{n}{4}}. \tag{II.145}$$

□

To get to the end of the proof, we only need to show that (II.133) holds, this comes from the following technical lemma.

Lemma 4.3. In the setup of Proposition 4.2, for all $R > 0$ and all sequence of $C^\infty(M)$ functions $(G_\alpha)_{\alpha\in\mathbb{N}}$ such that $\limsup_{\alpha\to\infty} |G_\alpha - 1| = 0$, there exists some $\varepsilon_\alpha > 0$ such that

$$\left\|(\Delta_g\varphi)\circ \exp_{x_\alpha} .G_\alpha - \Delta_\xi(\varphi\circ\exp_{x_\alpha})\right\|_{L^2(B_0(R\mu_\alpha))} \leq \varepsilon_\alpha \|\varphi\|_{H^2} \tag{II.146}$$

for all $\varphi \in C^\infty(M)$, with $\lim_{\alpha\to\infty} \varepsilon_\alpha = 0$.

Proof. We express our quantities in local coordinates (x^i). Since $g_\alpha = \exp_{x_\alpha}^* g$ is defined on $B_0(R\mu_\alpha)$ for a big enough α, we have $(g_\alpha(0))_{ij} = \delta_{ij}$ and $(\Gamma_\alpha(0))_{ij}^k = 0$ the connection coefficients, for $i, j, k = 1, \dots n$. Then

$$\begin{aligned} &\int_{B_0(R\mu_\alpha)} ((\Delta_g\varphi)\circ\exp_{x_\alpha} .G_\alpha - \Delta_\xi(\varphi\circ\exp_{x_\alpha}))^2 dx \\ &\quad= \int_{B_0(R\mu_\alpha)} \left(((g_\alpha)^{ij}G_\alpha - \delta^{ij})\partial_i\partial_j(\varphi\circ\exp_{x_\alpha}) + G_\alpha(\Gamma_\alpha)_{ij}^k\partial_k(\varphi\circ\exp_{x_\alpha})\right)^2 dx \\ &\quad\leq C\int_{B_0(R\mu_\alpha)} \left(|1-G_\alpha|\left|\nabla_\xi^2(\varphi\circ\exp_{x_\alpha})\right|^2 + |G_\alpha|^2|\Gamma|^2\left|\nabla_\xi(\varphi\circ\exp_{x_\alpha})\right|^2\right) dx \\ &\quad\leq \varepsilon_\alpha(R)\int_{B_{x_\alpha}(R\mu_\alpha)} \left(\left|\nabla^2\varphi\right|^2 + |\nabla_g\varphi|^2\right) dv_g \leq \varepsilon_\alpha \|\varphi\|_{H^2}, \end{aligned} \tag{II.147}$$

with $\varepsilon_\alpha \to 0$ uniformly with respect to R represents the supremum of the $C^\infty(M)$ functions that multiply $\nabla^j\varphi$, using that $g_\alpha = \xi + o(1)$ and $\Gamma = 0 + o(1)$ in a ball of radius $R\mu_\alpha$. □

Proof (of (II.133)). Applying this lemma on

$$\int_{B_{x_\alpha}(R\mu_\alpha)} \Delta_g u_\alpha \Delta_g\psi dv_g = \int_{B_0(R\mu_\alpha)} (\Delta_g u_\alpha)\circ\exp_{x_\alpha} .(\Delta_g\psi)\circ\exp_{x_\alpha} .\sqrt{|g_\alpha|}\, dx$$

by setting $G_\alpha = \sqrt{|g_\alpha|}$, $\varphi = u_\alpha$

$$= \int_{B_0(R\mu_\alpha)} \Delta_\xi (u_\alpha \circ \exp_{x_\alpha})(\Delta_g \psi) \circ \exp_{x_\alpha} dx + o(\|u_\alpha\|_{H^2} \|\psi\|_{H^2})$$

while by using $G_\alpha \equiv 1$, $\varphi = \psi$

$$= \int_{B_0(R\mu_\alpha)} \Delta_\xi (u_\alpha \circ \exp_{x_\alpha}) \Delta_\xi (\psi \circ \exp_{x_\alpha}) dx + o(\|u_\alpha\|_{H^2} \|\psi\|_{H^2})$$
$$= \int_{B_0(R)} \Delta_\xi \mathrm{b}(x) \Delta_\xi (\mu_\alpha^{\frac{n-4}{2}} \psi \circ \exp_{x_\alpha}(\mu_\alpha x)) dx + o(\|u_\alpha\|_{H^2} \|\psi\|_{H^2})$$

and the claim is proved. □

H^2 decomposition Theorem. We found a sequence of bounded functions in H^2-norm that are asymptotically critical for the chosen initial equation, and that actually weakly tends to 0 in H^2.

Definition 4.4. Let X be a Banach space, a sequence $(u_\alpha)_{\alpha\in\mathbb{N}}$ of functions in X is called a *Palais-Smale sequence* for the functional $J : X \to \mathbb{R}$ if $J(u_\alpha) = O(1)$ as $\alpha \to \infty$ and $\lim_{\alpha\to\infty} DJ(u_\alpha) = 0$ in the dual space.

Those type of sequences are of general interest in the study of partial differential equations, and give a new understanding to the notion of compactness of equations.

Definition 4.5. A functional J is said to satisfy the *Palais-Smale condition* if any Palais-Smale sequence $(u_\alpha)_{\alpha\in\mathbb{N}}$ for J admits a subsequence that converges strongly in $H^2(M)$.

What we now need to understand is when a given functional will satisfy this condition. One can cite Ambrosetti and Rabinowitz (1973), who provide results such as the Mountain Pass lemma that makes it possible to construct such sequences, and ultimately get solutions.

We saw in the first part of the section an example of a Palais-Smale sequence for

$$J(u) = \frac{1}{2} \int_M u P_g u dv_g - \frac{1}{2^\#} \int_M |u|^{2^\#} dv_g \tag{II.148}$$

in H^2 that does only weakly converge. The absence of compactness of the embedding of $H^2(M) \hookrightarrow L^{2^\#}(M)$ does not allow J to satisfy the Palais-Smale condition. However, in this case we exactly know in what way the sequences will not converge strongly. The following Theorem, which will be the basis for all the following study of these equations, will solely be presented and discussed, the interested reader can find a proof of it in Hebey and Robert (2001). It is a generalization to fourth-order equations of celebrated result by Struwe (see Struwe, 2008).

Theorem 4.6. Let (M,g) be a Riemannian manifold of dimension $n \geq 5$ and (u_α) a sequence of positive functions in $H^2(M)$ that are Palais-Smale for the functional J. We assume that $\exists\, C > 0$ such that $\|u_\alpha\|_{H^{k,p}} \leq C$. Then there exists a limit solution u_∞ to the equation (II.120), an integer $K \geq 1$, K converging sequences of points in M $(x_{i,\alpha})_\alpha$, and K sequences of positive real numbers $(\mu_{i,\alpha})_\alpha \to 0$ such that up to a subsequence,

$$u_\alpha = u_\infty + \sum_{i=1}^{K} B_\alpha^i + R_\alpha \tag{II.149}$$

with $R_\alpha \to 0$ in H^2 as $\alpha \to \infty$, $u_\alpha \rightharpoonup u_\infty$ in H^2 and

$$B^i_\alpha(x) = \eta_{x_{i,\alpha}}(x) \left(\frac{\mu_{i,\alpha}}{\mu^2_{i,\alpha} + d_g(x_{i,\alpha}, x)^2/\sqrt{\lambda_n}} \right)^{\frac{n-4}{2}} \tag{II.150}$$

are sequences of bubbles. Moreover, the energy splits, as

$$\begin{aligned} J(u_\alpha) &= J(u_\infty) + \sum_{i=1}^{K} J(B^i_\alpha) + o(1) \\ &= J(u_\infty) + K\left(\frac{2}{n} K_n^{-\frac{n}{4}}\right) + o(1) \end{aligned} \tag{II.151}$$

As we saw, the fact that all the bubbles tend weakly to 0 but keep a finite non-zero energy does not allow the Palais-Smale sequence (u_α), i.e. of asymptotically critical functions, to converge strongly to u_∞.

Remark : In this case, we are only interested in positive sequences of functions, but the Theorem actually holds with changing-sign functions. In that case, we can decompose

$$\begin{aligned} &u_\alpha = u_\infty + \sum_{i=1}^{K} u^i_{x_{i,\alpha},\mu_{i,\alpha}} + R_\alpha \qquad \text{with} \\ &u^i_{x_{i,\alpha},\mu_{i,\alpha}}(x) = \mu_{i,\alpha}^{-\frac{n-4}{2}} u^i(\mu_{i,\alpha}^{-1} \exp^{-1}_{x_{i,\alpha}}(x)), \end{aligned} \tag{II.152}$$

and the $u^i \in D^2(\mathbb{R}^n)$ are non-trivial solutions to the limiting critical equation in $\mathbb{R}^n$ with changing sign, $\Delta^2 u^i = \left|u^i\right|^{2^\#-2} u^i$ for $i = 1, \dots n$. Those functions can have much more complex behaviour than the bubbles, and less is known about changing-sign sequences. However, the study of those more general sequences is meaningful since we saw that retrieving positive solutions can sometimes be impossible in the fourth-order setting because of the absence of a comparison principle.

Moreover, this Theorem links the energy levels of a positive Palais-Smale sequence with the one of its weak limit. We know it must always differ by an integer multiple of the energy of a single bubble $\frac{2}{n} K_n^{-\frac{n}{4}}$. The consequence of this fact is that if the energy of the sequence always stays under this threshold, there simply cannot be a concentration bubble, and the sequence will strongly converge up to a subsequence in H^2.

We must also note that the energy split is not trivial. Actually, it can only be possible if the bubbles do not interact, and we have indeed the following result, still coming from Hebey and Robert (2001), that tells us this is the case.

Proposition 4.7.
(*Structure*) For any $i \neq j$,

$$\frac{d_g(x_{i,\alpha}, x_{j,\alpha})^2}{\mu_{i,\alpha}\mu_{j,\alpha}} + \frac{\mu_{i,\alpha}}{\mu_{j,\alpha}} + \frac{\mu_{j,\alpha}}{\mu_{i,\alpha}} \to \infty. \tag{II.153}$$

(*Estimate*) $\min_{i=1,\dots K} d_g(x_{i,\alpha}, x)\, |u_\alpha(x) - u_\infty(x)|^{\frac{2}{n-4}} \leq C$ for all α and all $x \in M$.

Sequences of solutions. Theorem 4.6 above applies to any Palais-Smale sequence of functions, but without further assumptions we are not able to give better estimates of the remainder term R_α. We know $\|R_\alpha\|_{H^2} \to 0$, but it could be unbounded and one cannot say anything about the pointwise behaviour of the sequence (u_α). In the following we will study sequences of solutions, and see how they can behave locally.

We saw a new definition of compactness particularly adapted for elliptical partial differential equations, that is the convergence, up to a subsequence, of Palais-Smale sequences. One could also quantify the lack of such compactness for the bi-Laplacian equations we are interested in. Our interest in the following will be the study of the stability of those equations, in a sense that will be precised. This discussion is inspired from the second-order setting exposed in Druet (2010), and from the introduction of Hebey and Robert (2011).

Example 4.8. Let us consider the function $f(x) = \frac{x}{1+x^2}$, one can aim to evaluate its critical points, that is x such that $f'(x) = 0$. In this case the set of critical points is $\mathrm{Crit}(f) = \{-1, 1\}$. By perturbing the equation for $\varepsilon > 0$ with $f_\varepsilon(x) = f(x) + \varepsilon \sin(x)$, one sees that $f_\varepsilon \to f$ in $C^1(\mathbb{R})$ as $\varepsilon \to 0$. However, every f_ε has critical points arbitrarily far from ± 1, that appear spontaneously when $\varepsilon \neq 0$.

Since f can be perturbed in C^1 in a way that creates new critical points that do not converge to the basis critical points, the equation $f'(x) = 0$ is said to be unstable.

This basic example, even if it is very far from our framework dealing with partial differential equations, allows to give a first insight on the stability we ask for our equation. Considering the very general equation on a compact manifold M

$$\Delta_g^2 u = f(u, x), \tag{II.154}$$

it will be said to be *stable* if a perturbation of f does not create new solutions that are far from the solutions to the initial equation. In other terms, it is stable if when $f_\varepsilon \to f$ in some space as $\varepsilon \to 0$, then the set of critical points $\mathrm{Crit}(f_\varepsilon) \to \mathrm{Crit}(f)$. We still need to specify the space of convergence of the perturbation, and what is meant by the set convergence.

Here we are interested in critical equations of the type

$$\Delta_g^2 u + b\Delta_g u + cu = u^{2^\# - 1} \tag{II.155}$$

where $\frac{b^2}{4} - c > 0$ to get the pointwise comparison principle. Coming from the conformal derivation of the equation, those can model the transformation of the Q-curvature when the metric g is Einstein. We consider furthermore only positive solutions. Such an equation is stable if and only if for any sequence of real numbers $(b_\alpha), (c_\alpha)$ converging respectively to b and c and any bounded sequence (u_α) in H^2 of smooth positive non-trivial solutions to

$$\Delta_g^2 u + b_\alpha \Delta_g u + c_\alpha u = u^{2^\# - 1}, \tag{II.156}$$

then $\exists\ u_\infty \in C^4(M)$ such that $u_\alpha \to u_\infty$ in $C^4(M)$.

Such a sequences of solutions are in particular Palais-Smale for the associated functional J defined previously (II.121), and can thus present bubbling. The question of analysing the stability of (II.155) is then strictly linked to the ability to describe the asymptotic behaviour of such sequences in better spaces that $H^2(M)$, and in particular to control the apparition of bubbles depending on the values of b and c. This will be the final goal of this work, discussed in Chapter IV.

A strong convergence Lemma. In the almost Euclidean case, we have a necessary condition for the formation of bubbles and the blowing-up of sequences of solutions, directly foreseen from the energy decomposition of Theorem 4.6.

Lemma 4.9. Let $\delta > 0$ and a smooth compact manifold of dimension $n \geq 5$, with a sequence of Riemannian metrics $(g_\alpha)_\alpha$ defined on the Euclidean ball $B_0(2\delta)$ such that $g_\alpha \to \xi$ the flat Euclidean metric in $C^4_{loc}(B_0(2\delta))$. Assume $(b_\alpha), (c_\alpha)$ are two bounded sequences of numbers such that $\frac{b_\alpha^2}{4} - c_\alpha > 0$ for all α, and $(w_\alpha)_\alpha$ is a sequence of bounded functions in $H^2(M)$ that are positive solutions to the equation

$$\Delta^2_{g_\alpha} w_\alpha + b_\alpha \Delta_{g_\alpha} w_\alpha + c_\alpha w_\alpha = w_\alpha^{2^\#-1} \tag{II.157}$$

in $B_0(2\delta)$. If $\|w_\alpha\|_{L^\infty(B_0(\delta))} \xrightarrow{\alpha\to\infty} \infty$ and $\|w_\alpha\|_{L^\infty(B_0(\delta)\setminus B_0(\delta/2))} \leq Cst$ for all α, then

$$\int_{B_0(\delta)} w_\alpha^{2^\#} dx \geq (1 + o(1)) K_n^{-\frac{n}{4}} \tag{II.158}$$

where K_n is the optimal constant for the Sobolev embedding as defined and discussed in 3.11.

Proof. We take $x_\alpha \in B_0(\delta)$ such that $w_\alpha(x_\alpha) = \max_{x\in\overline{B_0(\delta)}} w_\alpha(x)$, and define $\nu_\alpha^{\frac{4-n}{2}} := w_\alpha(x_\alpha)$. By hypothesis, we have $\nu_\alpha \to 0$ and one can suppose up to taking a large enough α that $x_\alpha \in B_0(\delta/2)$. We write $\tilde{w}_\alpha(x) := \nu_\alpha^{\frac{n-4}{2}} w_\alpha(x_\alpha + \nu_\alpha x)$ a rescaled version of the sequence of functions we are interested in, such that their maximum always happens at $x = 0$ and is equal to 1.

Let $R > 1$, for a large enough α, $\tilde{w}_\alpha$ will be defined on $B_0(R)$, and since we assume $\frac{b_\alpha^2}{4} - c_\alpha > 0$, we can define

$$d_{1,\alpha} := \frac{b_\alpha}{2} + \sqrt{\frac{b_\alpha^2}{4} - c_\alpha} \qquad\qquad d_{2,\alpha} := \frac{b_\alpha}{2} - \sqrt{\frac{b_\alpha^2}{4} - c_\alpha} \tag{II.159}$$

following from the Corollary 3.8.i. Then we can verify

$$\begin{aligned} \tilde{w}_\alpha^{2^\#-1} = \nu_\alpha^{\frac{n+4}{2}} w_\alpha^{2^\#-1} &= \nu_\alpha^{\frac{n+4}{2}} (\Delta^2_{g_\alpha} w_\alpha + b_\alpha \Delta_{g_\alpha} w_\alpha + c_\alpha w_\alpha) \\ &= \Delta^2_{\tilde{g}_\alpha} \tilde{w}_\alpha + b_\alpha \nu_\alpha^2 \Delta_{\tilde{g}_\alpha} \tilde{w}_\alpha + c_\alpha \nu_\alpha^4 \tilde{w}_\alpha \\ &= \left(\Delta_{\tilde{g}_\alpha} + d_{1,\alpha}\nu_\alpha^2\right) \circ \left(\Delta_{\tilde{g}_\alpha} + d_{2,\alpha}\nu_\alpha^2\right) \tilde{w}_\alpha, \end{aligned} \tag{II.160}$$

this equation holding in $B_0(R)$ for all $\alpha \gg 1$ large enough, setting $\tilde{g}_\alpha(x) := g_\alpha(x_\alpha + \nu_\alpha x)$. But now since the $\tilde{w}_\alpha$ are bounded, with precisely $0 \leq \tilde{w}_\alpha \leq 1$, then there exists $\tilde{w} \in C^4(\mathbb{R}^n)$ such that $\tilde{w}_\alpha \to \tilde{w}$ in $C^4_{loc}(\mathbb{R}^n)$ for $\alpha \to \infty$ thanks to elliptic theory.

Using the invariance of the norms under the scaling operation, we have also $\tilde{w} \in H^2(\mathbb{R}^n)$, and by taking the limit in the equation (II.160), $\tilde{w}$ solves

$$\Delta^2 \tilde{w} = \tilde{w}^{2^\#-1}. \tag{II.161}$$

As in (II.110), by Lin's classification we then know that $\tilde{w}$ is given by (II.112), and attains the optimal Sobolev constant. Then for all $R > 1$,

$$\int_{B_0(R)} \tilde{w}_\alpha^{2^\#} dv_{\tilde{g}_\alpha} = \int_{B_{x_\alpha}(R\nu_\alpha)} w_\alpha^{2^\#} dv_{g_\alpha} \leq (1 + o(1)) \int_{B_0(\delta)} w_\alpha^{2^\#} dx \tag{II.162}$$

by the assumption that $\|w_\alpha\|_{L^\infty(B_0(\delta)\setminus B_0(\delta/2))} \leq Cst$. Now since $\tilde{w}_\alpha \to \tilde{w}$ in $C^4_{loc}(\mathbb{R}^n)$ and thus also in $L^\infty_{loc}(\mathbb{R}^n)$, by Fatou's Lemma,

$$\liminf_{\alpha\to\infty} \int_{B_0(\delta)} w_\alpha^{2^\#} dx \geq \liminf_{\alpha\to\infty} \int_{B_0(R)} \tilde{w}_\alpha^{2^\#} dv_{\tilde{g}_\alpha} \geq \int_{B_0(R)} \tilde{w}^{2^\#} dx. \tag{II.163}$$

Letting $R \to \infty$, and since $\tilde{w}$ reaches the constant, we obtain

$$\liminf_{\alpha\to\infty} \int_{B_0(\delta)} w_\alpha^{2^\#} dx \geq K_n^{-n/4}. \tag{II.164}$$

□

Remark : The tools we used in the proof of this Lemma are standard and will be used throughout the rest of this Thesis. This result will be especially useful to us in its contrapositive formulation; that is, if we are able to prove that

$$\limsup_{\alpha\to\infty} \int_{B_0(\delta)} w_\alpha^{2^\#} dx \xrightarrow{\delta\to 0} 0, \tag{II.165}$$

i.e. that the sequence of functions does not concentrate in $L^{2^\#}$-norm, by this Lemma we will know that we have the strong convergence $w_\alpha \to w_\infty$ in $C^4_{loc}(B_0(\delta))$.

Unlike Theorem 4.6 which gives a global decomposition in $H^2(M)$, this result is local, and is a sequential version of a ε-regularity result.

Chapter III

Point-wise estimates for blow-up

In this chapter, we finally dive in the main problem our research. Using all the knowledge we acquired on the study of critical bi-Laplacian equations arising from conformal transformations in a compact Riemannian manifold, the aim is to study with more details the asymptotic behaviour of sequences of solutions to equations such as those discussed in the previous sections. With this chapter, we also dive in the main reference of this work, that is the article *asymptotic analysis for fourth-order Paneitz equations with critical growth* (Hebey & Robert, 2011). Throughout the rest of the work all the propositions will come from this article, unless specified otherwise.

1 Firsts results

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$. The purpose of this section will be to better understand the blowing-up behavior of solutions to Paneitz-type equations. Let b, c be positive real numbers such that $\frac{b^2}{4} - c > 0$, and $(b_\alpha), (c_\alpha)$ be converging sequences of real numbers with respective limits b and c for $\alpha \to \infty$, we want to study the behaviour of sequences (u_α) of positive non-trivial solutions to the equations

$$\Delta_g^2 u + b_\alpha \Delta_g u + c_\alpha u = u^{2^\# - 1}, \tag{III.1}$$

bounded in H^2 and blowing up

$$\|u_\alpha\|_{L^\infty} \to +\infty. \tag{III.2}$$

This last assumption shows that $(u_\alpha)_\alpha$ does not converge strongly in $H^2(M)$.

As we already stated, there are some known results resembling those of the Yamabe-type equation, that allow us to decompose u_α in a finite sum of bubbles :

$$u_\alpha = u_\infty + \sum_{i=1}^{K} B_\alpha^i + R_\alpha \tag{III.3}$$

with $R_\alpha \to 0$ in H^2 as $\alpha \to \infty$.

Since it will be useful to understand the behaviour of u_α around the blowing-up points, we recall the rescaling operation

$$R_x^\mu u(y) := \mu^{\frac{n-4}{2}} u(\exp_x(\mu y)) \tag{III.4}$$

for all $y \in \mathbb{R}^n$ such that $\mu\,|y| < i_g$, with $x \in M$, $\mu > 0$ and a smooth function $u : M \to \mathbb{R}$. We also realise that the blowing-up bubble sequences are more or less just a rescaling of the basic euclidean bubble b, solution of the critical equation

$$\Delta^2 u = u^{2^\# - 1} \qquad \text{on } \mathbb{R}^n, \tag{III.5}$$

and we have $R^{\mu_{i,\alpha}}_{x_{i,\alpha}} B^i_\alpha(x) = \eta(|\mu_{i,\alpha} x|)\mathrm{b}(x)$, $i = 1, \dots K$. In the bubble on M we had to add the cut-off function so that the local definition of b could be extended on the whole manifold. Furthermore, following from the structure equation (II.153) we have the following property.

Lemma 1.1. Let us define the set I_i of the indices $j = 1, \dots K$ for which $d_g(x_{i,\alpha}, x_{j,\alpha}) = \mathcal{O}(\mu_{i,\alpha})$ and $\mu_{j,\alpha} = o(\mu_{i,\alpha})$, i.e. the cooncentration points that will not be sent to infinity and whose bubble concentrate faster than B^i_α. We also define

$$\mathcal{S}_i := \{\lim_{\alpha\to\infty} \frac{1}{\mu_{i,\alpha}} \exp^{-1}_{x_{i,\alpha}}(x_{j,\alpha}),\ j \in I_i\} \subset \mathbb{R}^n, \tag{III.6}$$

then

$$R^{\mu_{i,\alpha}}_{x_{i,\alpha}} B^j_\alpha \to 0 \tag{III.7}$$

in $L^\infty_{loc}(\mathbb{R}^n \setminus \mathcal{S}_i)$.

From the estimate we have in the Proposition II.4.7, let us define $\mathcal{S} := \{\lim_{\alpha\to\infty} x_{i,\alpha}\}$ the set of the limits of the blowing-up points, then $u_\alpha \to u_\infty$ in $C^4_{loc}(M \setminus \mathcal{S})$. The Lemma above tells us that when we zoom on a singularity and re-normalize the bubble, all other bubbles tend to 0 in the local map in $\mathbb{R}^n$, except for those in I_i. Since $y \in \mathcal{S}_i$ means $R^{\mu_{i,\alpha}}_{x_{i,\alpha}} u_a(y) = u_\alpha(x_{j,\alpha})$ for some $j \in I_i$, y is then the center of those functions that do not tend to zero in the local map, we could say that these bubbles are accumulated on B^i_α.

The following proposition tells us that u_α, that decomposes as a sum of bubbles in $H^2(M)$, looks almost exaclty like a single ground state bubble when we apply the same rescaling around another concentration point.

Proposition 1.2. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, $b, c > 0$ positive real numbers such that $\frac{b^2}{4} - c > 0$. Let $(b_\alpha), (c_\alpha)$ be two sequences of real numbers converging to b and c respectively, and (u_α) a bounded sequence of functions in H^2 that are positive non-trivial solutions to the equation (III.1) that satisfies (III.2). Then up to a subsequence, for i fixed,

$$R^{\mu_{i,\alpha}}_{x_{i,\alpha}} u_\alpha \to \mathrm{b} \tag{III.8}$$

in $C^4_{loc}(\mathbb{R}^n \setminus \mathcal{S}_i)$.

Proof. We already know that $R^{\mu_{i,\alpha}}_{x_{i,\alpha}} B^j_\alpha \to 0$ in $L^\infty_{loc}(\mathbb{R}^n \setminus \mathcal{S}_i)$ if $i \neq j$, and we have also

$$\begin{aligned} &R^{\mu_{i,\alpha}}_{x_{i,\alpha}} B^i_\alpha(x) = \eta(|\mu_{i,\alpha} x|)\mathrm{b}(x), \text{ so that} \\ &R^{\mu_{i,\alpha}}_{x_{i,\alpha}} \sum_{j=1}^K B^j_\alpha \to \mathrm{b} \quad \text{in } L^\infty_{loc}(\mathbb{R}^n \setminus \mathcal{S}_i). \end{aligned} \tag{III.9}$$

We also know by the decomposition Theorem 4.6, that up to a subsequence

$$R^{\mu_{i,\alpha}}_{x_{i,\alpha}} u_\alpha - R^{\mu_{i,\alpha}}_{x_{i,\alpha}} \sum_{j=1}^{K} B^j_\alpha \to 0 \tag{III.10}$$

in H^2, so also in $L^{2^\#}_{loc}(\mathbb{R}^n)$ as $\alpha \to \infty$. We define $\tilde{u}_\alpha := R^{\mu_{i,\alpha}}_{x_{i,\alpha}} u_\alpha$, it solves the equation

$$\Delta^2_{g_\alpha} \tilde{u}_\alpha + b_\alpha \mu^2_{i,\alpha} \Delta_{g_\alpha} \tilde{u}_\alpha + c_\alpha \mu^4_{i,\alpha} \tilde{u}_\alpha = \tilde{u_\alpha}^{2^\#-1} \tag{III.11}$$

where $g_\alpha(x) := \left(\exp^*_{x_{i,\alpha}} g\right)(\mu_{i,\alpha} x)$. Note that $g_\alpha \to \xi$, the flat metric, in $C^4_{loc}(\mathbb{R}^n)$ as $\alpha \to \infty$.

We want to show that $(\tilde{u}_\alpha)$ is a bounded sequence of $L^\infty_{loc}(\mathbb{R}^n \setminus \mathcal{S}_i)$, then it will converge strongly by elliptic theory. To do that, we start with the easier case where we exclude every center, then it is a simple calculation. Writing

$$\tilde{\mathcal{S}}_i := \{ \lim_{\alpha \to \infty} \frac{1}{\mu_{i,\alpha}} \exp_{x_{i,\alpha}}(x_{j,\alpha}),\ j = 1, \dots K \}, \tag{III.12}$$

we fix $K \subset\subset \mathbb{R}^n \setminus \tilde{\mathcal{S}}_i$, then for all $j \neq i$ and $y \in K$

$$\begin{aligned} d_g(x_{j,\alpha}, \exp_{x_{i,\alpha}}(\mu_{i,\alpha} y)) &\geq C \left| \exp^{-1}_{x_{i,\alpha}}(x_{j,\alpha}) - \mu_{i,\alpha} y \right| \\ &\geq C\mu_{i,\alpha} \left| \frac{1}{\mu_{i,\alpha}} \exp^{-1}_{x_{i,\alpha}}(x_{j,\alpha}) - y \right| \geq C\mu_{i,\alpha}, \end{aligned} \tag{III.13}$$

so that

$$\begin{aligned} \left| R^{\mu_{i,\alpha}}_{x_{i,\alpha}} u_\alpha(y) \right| &= \left| \mu_{i,\alpha}^{\frac{n-4}{2}} u_\alpha(\exp_{x_{i,\alpha}}(\mu_{i,\alpha} y)) \right| \\ &\leq \mu_{i,\alpha}^{\frac{n-4}{2}} \left(\left| u_\alpha(\exp_{x_{i,\alpha}}(\mu_{i,\alpha} y)) - u_\infty(\exp_{x_{i,\alpha}}(\mu_{i,\alpha} y)) \right| + \|u_\infty\|_{L^\infty} \right) \\ &\leq C\mu_{i,\alpha}^{\frac{n-4}{2}} d_g(x_{j,\alpha}, \exp_{x_{i,\alpha}}(\mu_{i,\alpha} y))^{-\frac{n-4}{2}} \\ &\leq C \end{aligned} \tag{III.14}$$

using the estimate of Proposition II.4.7 at $x = \exp_{x_{i,\alpha}}(\mu_{i,\alpha} y)$. The problem is that we would want to extend the point-wise bound on $R^{\mu_{i,\alpha}}_{x_{i,\alpha}} u_\alpha(y)$ to the centers of the bubbles with whom we know there should not be any problems, since they tend to 0 in the local chart. We also would like to be able to estimate the rescaled $\tilde{u}_\alpha$ at its very own center, which obviously is in $\tilde{\mathcal{S}}_i$ but not in $\mathcal{S}_i$, it will be fundamental in the following results.

We use the fact that, for $x \in \mathbb{R}^n \setminus \mathcal{S}_i$ and a small enough $\delta > 0$ so that $B_x(\delta) \subset \mathbb{R}^n \setminus \mathcal{S}_i$,

$$\begin{aligned} \limsup_{\alpha \to \infty} \int_{B_x(\delta)} \tilde{u}_\alpha^{2^\#}\ dx &= \limsup_{\alpha \to \infty} \int_{B_x(\delta)} R^{\mu_{i,\alpha}}_{x_{i,\alpha}} \left(\sum_{j=1}^{K} B^j_\alpha \right)^{2^\#} dx \\ &\leq \int_{B_x(\delta)} \mathrm{b}^{2^\#}\ dx \xrightarrow{\delta \to 0} 0 \end{aligned} \tag{III.15}$$

using the well known Fatou's Lemma and the fact that b is a nice smooth standard function. The rescaled version of u_α does not concentrate locally in $K \subset\subset \mathcal{S}_i$, so by the Lemma II.4.9 we know that

$(\tilde{u}_\alpha)$ does converge strongly in $C^4(K)$. Finally the latter result provides the conclusion $R_{x_{i,\alpha}}^{\mu_{i,\alpha}} u_\alpha = \tilde{u}_\alpha \to \mathrm{b}$ as $\alpha \to \infty$ in $C^4_{loc}(\mathbb{R}^n \setminus \mathcal{S}_i)$ up to a subsequence. □

We now want to refine the estimate we had on the remainder term of the decomposition (III.3), writing for all $x \in M$

$$r_\alpha(x) := \min_{i=1,\dots K} d_g(x_{i,\alpha}, x), \tag{III.16}$$

we had in II.4.7 that $r_\alpha^4 \, |u_\alpha - u_\infty|^{2^\#-2} \leq C$, here we will show that substracting the contribution of the bubbles allows us to make the estimate converge to 0.

Proposition 1.3. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, $b, c > 0$ positive real numbers such that $\frac{b^2}{4} - c > 0$. Taking $(b_\alpha), (c_\alpha)$ sequences of real numbers converging to b and c respectively, and (u_α) a bounded sequence of functions in H^2 that are positive solutions to the equation (III.1) that satisfy (III.2). Then up to a subsequence,

$$r_\alpha^4 \Big| u_\alpha - u_\infty - \sum_{i=1}^K B_\alpha^i \Big|^{2^\#-2} \to 0 \qquad \text{in } L^\infty(M). \tag{III.17}$$

Actually in the following, we will prove the stronger result that $D_\alpha^4 \big| u_\alpha - u_\infty - \sum_{i=1}^K B_\alpha^i \big|^{2^\#-2} \to 0$ in $L^\infty(M)$ as $\alpha \to \infty$, where

$$D_\alpha(x) := \min_{i=1,\dots K} \left(d_g(x_{i,\alpha}, x) + \mu_{i,\alpha} \right). \tag{III.18}$$

To do that we will focus on the maximising point $x_\alpha \in M$ such that

$$D_\alpha(x_\alpha)^4 \Big| u_\alpha(x_\alpha) - u_\infty(x_\alpha) - \sum_{i=1}^K B_\alpha^i(x_\alpha) \Big|^{2^\#-2} = \max_{x \in M} D_\alpha(x)^4 \Big| u_\alpha(x) - u_\infty(x_\alpha) - \sum_{i=1}^K B_\alpha^i(x) \Big|^{2^\#-2}. \tag{III.19}$$

We will need the following Lemma.

Lemma 1.4. In the same setup that the Proposition 1.3, with D_α defined as in (III.18) and x_α as in (III.19), if there is $\varepsilon_0 > 0$ and a i_0 such that

$$D_\alpha(x_\alpha)^4 B_\alpha^{i_0}(x_\alpha)^{2^\#-2} \geq \varepsilon_0 \text{ for all } \alpha, \tag{III.20}$$

then

$$D_\alpha(x_\alpha)^4 \Big| u_\alpha(x_\alpha) - u_\infty(x_\alpha) - \sum_{i=1}^K B_\alpha^i(x_\alpha) \Big|^{2^\#-2} \to 0. \tag{III.21}$$

Proof. Using that $2^{\#}-2=\frac{8}{n-4}$, the assumption (III.20) reads

$$\eta\left(d_g(x_{i_0,\alpha},x_\alpha)\right)\left(\frac{D_\alpha(x_\alpha)\mu_{i_0,\alpha}}{\mu_{i_0,\alpha}^2+d_g(x_{i_0,\alpha},x_\alpha)^2/\sqrt{\lambda_n}}\right)^4\geq\varepsilon_0\quad\forall\ \alpha. \tag{III.22}$$

The only way this inequality can hold is if $\frac{d_g(x_{i_0,\alpha},x_\alpha)}{\mu_{i_0,\alpha}}$ stays bounded and $\frac{D_\alpha(x_\alpha)}{\mu_{i_0,\alpha}}\geq\delta(\varepsilon_0)$, $\delta>0$ being a constant depending on ε_0 whose exact expression does not change our arguments.

In other words, since $\mu_{i_0,\alpha}\to 0$, we need that

$$\begin{cases} d_g(x_{i_0,\alpha},x_\alpha)\to 0,\\ \dfrac{d_g(x_{i_0,\alpha},x_\alpha)}{\mu_{i_0,\alpha}}\to\lambda\geq 0 \text{ a constant, and}\\ \dfrac{\mu_{j,\alpha}}{\mu_{i_0,\alpha}}+\dfrac{d_g(x_{j,\alpha},x_\alpha)}{\mu_{i_0,\alpha}}\geq\delta \text{ for all } j \text{ and all } \alpha. \end{cases} \tag{III.23}$$

Taking $y_\alpha=\frac{1}{\mu_{i_0,\alpha}}\exp_{x_{i_0,\alpha}}^{-1}(x_\alpha)\in\mathbb{R}^n$ the local image of x_α around $B_\alpha^{i_0}$, it comes by simple calculation from our conditions that $d_g(y_\alpha,\mathcal{S}_{i_0})\geq\varepsilon$ independent of α. In that case the Proposition 1.2 gives us

$$\begin{aligned} &R_{x_{i_0,\alpha}}^{\mu_{i_0,\alpha}}u_\alpha(y_\alpha)-\mathrm{b}(y_\alpha)\to 0\\ \Leftrightarrow\ &\mu_{i_0,\alpha}^{\frac{n-4}{2}}\left(u_\alpha(x_\alpha)-\frac{1}{\eta(d_g(x_{i_0,\alpha},x_\alpha))}B_\alpha^{i_0}(x_\alpha)\right)\to 0. \end{aligned} \tag{III.24}$$

By the second condition of (III.23), we also know that $|y_\alpha|\leq C$ also independently of α and that

$$\begin{aligned} D_\alpha(x_\alpha)&=\min_{i=1,\dots K}\left(d_g(x_{i,\alpha},x_\alpha)+\mu_{i,\alpha}\right)\leq\mu_{i_0,\alpha}\left(\frac{d_g(x_{i_0,\alpha},x_\alpha)}{\mu_{i_0,\alpha}}+1\right)\\ &\leq C\mu_{i_0,\alpha}. \end{aligned} \tag{III.25}$$

Remembering that the structure Equation (II.153) tells us that since $d_g(x_{i_0,\alpha},x_\alpha)\to 0$, for all $j\neq i_0$ we have $\mu_{i_0,\alpha}^{\frac{n-4}{2}}B_\alpha^j(x_\alpha)\to 0$, we can put everything together and finally get

$$\begin{aligned} &D_\alpha(x_\alpha)^4\left|u_\alpha(x_\alpha)-u_\infty(x_\alpha)-\sum_{i=1}^K B_\alpha^i(x_\alpha)\right|^{2^{\#}-2}\\ &\qquad\leq C\left|\mu_{i_0,\alpha}^{\frac{n-4}{2}}u_\alpha(x_\alpha)-\mu_{i_0,\alpha}^{\frac{n-4}{2}}u_\infty(x_\alpha)-\mu_{i_0,\alpha}^{\frac{n-4}{2}}\sum_{i=1}^K B_\alpha^i(x_\alpha)\right|^4\to 0 \end{aligned} \tag{III.26}$$

□

Proof (of Proposition 1.3). By contradiction, we suppose that there is $\varepsilon_0\geq 0$ such that

$$D_\alpha(x_\alpha)^4\left|u_\alpha(x_\alpha)-u_\infty(x_\alpha)-\sum_{i=1}^K B_\alpha^i(x_\alpha)\right|^{2^{\#}-2}\geq\varepsilon_0, \tag{III.27}$$

by contrapositive of the previous Lemma we must have $\lim_{\alpha\to\infty}D_\alpha(x_\alpha)^4B_\alpha^i(x_\alpha)^{2^{\#}-2}=0$ for all i. Let us prove that $u_\alpha(x_\alpha)\to\infty$, i.e. that the maximal value of the estimate is reached at a concentration point of u_α.

We know that $D_\alpha(x_\alpha)$ is bounded, let us pretend that up to a subsequence $D_\alpha(x_\alpha) \to \delta > 0$. But then we must have $B^i_\alpha(x_\alpha) \to 0$ for all i to satisfy the limit, and in a neighborhood of the maximum x_α

$$\begin{aligned} \varepsilon_0 &\leq D_\alpha(x)^4 \big| u_\alpha(x) - u_\infty(x) - \sum_{i=1}^K B^i_\alpha(x) \big|^{2^\#-2} \\ &\leq C \left| u_\alpha(x) - u_\infty(x) \right|^{2^\#-2} + o(1). \end{aligned} \tag{III.28}$$

If $u_\alpha(x_\alpha)$ does not tend to infinity, in a neighborhood of x_a, u_α is uniformly bounded and thus by elliptic theory $u_\alpha \to u_\infty$ in $L^\infty(\Omega)$ where Ω is a neighborhood of the limit of x_α. But then in Ω, $u_\alpha(x) - u_\infty(x) \to 0$ and that contradicts (III.28), so that we know $u_\alpha(x_\alpha) \to \infty$.

On the other hand if $D_\alpha(x_\alpha) \to 0$, by (III.27) and $\lim_{\alpha\to\infty} D_\alpha(x_\alpha)^4 B^i_\alpha(x_\alpha)^{2^\#-2} = 0$ for all i we immediately must have $u_\alpha(x_\alpha) \to \infty$.

Now that we know that x_α tends to the center of a bubble of u_α, as in the proof of Proposition 1.2 we will rescale u_α around it, we define $\tilde{u}_\alpha := R^{\mu_\alpha}_{x_\alpha} u_\alpha$, with μ_α given by $\mu_\alpha^{2-\frac{n}{2}} = u_\alpha(x_\alpha)$ so that first $\tilde{u}_\alpha(0) = 1$ and

$$\Delta^2_{g_\alpha} \tilde{u}_\alpha + b_\alpha \mu^2_\alpha \Delta_{g_\alpha} \tilde{u}_\alpha + c_\alpha \mu^4_\alpha \tilde{u}_\alpha = \tilde{u}_\alpha^{2^\#-1}, \tag{III.29}$$

writing $g_\alpha = (\exp^*_{x_\alpha} g)(\mu_\alpha x)$. Since $u_\alpha(x_\alpha) \to \infty$ we know $\mu_\alpha \to 0$ and thus $g_\alpha \to \xi$ in $C^4_{loc}(\mathbb{R}^n)$. Because the u_α are a bounded sequence of $H^2(M)$, up to a subsequence $\tilde{u}_\alpha \rightharpoonup \tilde{u}_\infty$ in H^2_{loc}, and $\tilde{u}_\infty$ must verify the limit equation

$$\Delta^2 \tilde{u}_\infty = \tilde{u}_\infty^{2^\#-1} \qquad \text{in } \mathbb{R}^n. \tag{III.30}$$

We want to conclude about the strong convergence of $\tilde{u}_\alpha$ punctually, let us show that it is uniformly bounded on a neighborhood of 0. Still as in the proof of the previous Proposition, we define

$$\begin{aligned} &\tilde{\mathcal{S}} := \{ \lim_{\alpha\to\infty} \frac{1}{\mu_\alpha} \exp^{-1}_{x_\alpha}(x_{i,\alpha}),\ i \in \tilde{I}\}, \text{ where} \\ &\tilde{I} := \{ i = 1 \dots K \quad \text{s.t. } d_g(x_{i,\alpha}, x_\alpha) = \mathcal{O}(\mu_\alpha) \text{ and } \mu_{i,\alpha} = o(\mu_\alpha)\} \end{aligned} \tag{III.31}$$

the equivalent of $\mathcal{S}_i$, and fix $K \subset\subset \mathbb{R}^n \setminus \tilde{\mathcal{S}}$. Since

$$D_\alpha(x_\alpha)^4 B^i_\alpha(x_\alpha)^{2^\#-2} \to 0 \quad \forall\ i, \tag{III.32}$$

by the definition of x_α

$$\begin{aligned} D_\alpha(x)^4 \big| u_\alpha(x) - u_\infty(x) - \sum_{i=1}^K B^i_\alpha(x) \big|^{2^\#-2} &\leq D_\alpha(x_\alpha)^4 \big| u_\alpha(x_\alpha) - u_\infty(x_\alpha) - \sum_{i=1}^K B^i_\alpha(x_\alpha) \big|^{2^\#-2} \\ &\leq D_\alpha(x_\alpha)^4 \left| u_\alpha(x_\alpha) - u_\infty(x_\alpha) \right|^{2^\#-2} + o(1). \end{aligned} \tag{III.33}$$

Then estimating this at $x = \exp_{x_\alpha}(\mu_\alpha y)$, $y \in K$, and multiplying by μ_α^4, we get

$$\begin{aligned} D_\alpha(x)^4 \Big| \tilde{u}_\alpha(y) - \mu_\alpha^{\frac{n-4}{2}} u_\infty(x) - \mu_\alpha^{\frac{n-4}{2}} \sum_{i=1}^K B_\alpha^i(x) \Big|^{2^\#-2} & \\ \leq D_\alpha(x_\alpha)^4 \mu_\alpha^4 \Big| u_\alpha(x_\alpha) - u_\infty(x_\alpha) - \sum_{i=1}^K B_\alpha^i(x_\alpha) \Big|^{2^\#-2} & \\ \leq D_\alpha(x_\alpha)^4 (1 + o(1)) + o(1). & \end{aligned} \tag{III.34}$$

using that $u_\alpha(x_\alpha)^{2^\#-2} = \mu_\alpha^{-4}$ and that $u_\alpha(x_\alpha) \to \infty$. Finally since $\mu_\alpha \to 0$ and $R_{x_\alpha}^{\mu_\alpha} B_\alpha^i(y) \to 0$ in $\mathbb{R}^n \setminus \tilde{\mathcal{S}}$ we conclude

$$\tilde{u}_\alpha^{2^\#-2} \leq \left(\frac{D_\alpha(x_\alpha)}{D_\alpha(x)} \right)^4 (1 + o(1)) + o(1). \tag{III.35}$$

But $y \in K$, so that $D_\alpha(x_\alpha) = \mathcal{O}(D_\alpha(x))$ and so for any $K \subset\subset \mathbb{R}^n \setminus \tilde{\mathcal{S}}$, $\tilde{u}_\alpha \leq C$. This gives us $\tilde{u}_\alpha \to u_\infty$ strongly in $C^4_{loc}(\mathbb{R}^n \setminus \tilde{\mathcal{S}})$.

We end the proof by noticing that $0 \notin \tilde{\mathcal{S}}$, otherwise there would exist i such that $x_\alpha = x_{i,\alpha}$ and $\mu_{i,\alpha} = o(\mu_\alpha)$, which would not give us the initial inequality (III.27). Since $\tilde{u}_\alpha(0) = 1$, using the strong convergence in $\mathbb{R}^n \setminus \tilde{\mathcal{S}}$, we know that $\tilde{u}_\infty(0) = 1$, and so $\tilde{u}_\infty \not\equiv 0$. But $\tilde{u}_\infty$ solves the Equation (III.30) for which we know all the solutions (Lin, 1998), there must be $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ such that

$$\tilde{u}_\infty(x) = \left(\frac{\lambda}{\lambda^2 + \frac{|x - x_0|^2}{\sqrt{\lambda_n}}} \right)^{\frac{n-4}{2}} \tag{III.36}$$

with $\lambda(\lambda - 1)\sqrt{\lambda_n} + |x_0| = 0$.

Finally, by the decomposition in $L^{2^\#}$, applying the rescaling we got

$$\tilde{u}_\alpha = \mu_\alpha^{\frac{n-4}{2}} u_\infty(\exp_{x_\alpha}(\mu_\alpha \, \cdot)) + \mu_\alpha^{\frac{n-4}{2}} \sum_{i=1}^K B_\alpha^i(\exp_{x_\alpha}(\mu_\alpha \, \cdot)) + \mu_\alpha^{\frac{n-4}{2}} R_\alpha \to 0 \quad \text{in } L^{2^\#}. \tag{III.37}$$

By strong convergence in any $K \subset\subset \mathbb{R}^n \setminus \tilde{\mathcal{S}}$, we should also have

$$\int_K \tilde{u}_\infty^{2^\#} \, dx = 0, \tag{III.38}$$

which contradicts the form of $\tilde{u}_\infty$. We finally arrived at a contradiction, we must have

$$D_\alpha^4 \Big| u_\alpha - u_\infty - \sum_{i=1}^K B_\alpha^i \Big|^{2^\#-2} \to 0 \quad \text{in } L^\infty(M) \text{ as } \alpha \to \infty \tag{III.39}$$

□

Corollary 1.5. Under the same assumptions as in the Proposition 1.3, there exists $C \geq 0$ such that up to a subsequence

$$r_\alpha^{\frac{n-4}{2}} u_\alpha \leq C \quad \text{in } L^\infty(M) \tag{III.40}$$

Proof. We have

$$\begin{aligned} r_\alpha^{\frac{n-4}{2}} u_\alpha &\leq r_\alpha^{\frac{n-4}{2}} \left| u_\alpha - u_\infty - \sum_{i=1}^K B_\alpha^i \right| + r_\alpha^{\frac{n-4}{2}} |u_\infty| + r_\alpha^{\frac{n-4}{2}} \sum_{i=1}^K B_\alpha^i \\ &\leq r_\alpha^{\frac{n-4}{2}} |u_\infty| + \sum_{i=1}^K d_g(x_{i,\alpha}, \cdot)^{\frac{n-4}{2}} B_\alpha^i + o(1). \end{aligned} \tag{III.41}$$

But on one hand u_∞ is bounded, and since M is compact r_α is too, and on the other, writing $d_{i,\alpha}(x) := d_g(x_{i,\alpha}, x)$ to ease the calculations, for $i = 1, \dots K$

$$\begin{aligned} d_{i,\alpha}^{\frac{n-4}{2}} B_\alpha^i &= d_{i,\alpha}^{\frac{n-4}{2}} \mu_{i,\alpha}^{-\frac{n-4}{2}} \left(1 + \left(\frac{d_{i,\alpha}}{\mu_{i,\alpha}} \right)^2 \frac{1}{\sqrt{\lambda_n}} \right)^{-\frac{n-4}{2}} \\ &= \left(\frac{d_{i,\alpha}}{\mu_{i,\alpha}} \left(1 + \left(\frac{d_{i,\alpha}}{\mu_{i,\alpha}} \right)^2 \frac{1}{\sqrt{\lambda_n}} \right)^{-1} \right)^{\frac{n-4}{2}} \leq C \end{aligned} \tag{III.42}$$

whatever the value of $\frac{d_{i,\alpha}}{\mu_{i,\alpha}}$. That concludes the proof. □

2 Estimates for higher order derivatives

With the previous result, we know that $u_\alpha \leq C r_\alpha^{-\frac{n-4}{2}}$, but when we get closer to a concentration point this estimate becomes less and less accurate. The purpose of this section is to get sharper inequalities to know the speed of growth of u_α and its derivatives around the blowing-up points. The main result will then be to prove the following.

Theorem 2.1. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, $b, c > 0$ positive real numbers such that $\frac{b^2}{4} - c > 0$. Taking $(b_\alpha), (c_\alpha)$ sequences of real numbers converging to b and c respectively, and (u_α) a bounded sequence of functions in H^2 that are positive solutions to the equation

$$\Delta_g^2 u + b_\alpha \Delta_g u + c_\alpha u = u^{2^\# - 1}, \tag{III.43}$$

such that $\|u_\alpha\|_{L^\infty} \to +\infty$. Then there exists $C > 0$ such that up to a subsequence

$$\left| \nabla^j u_\alpha \right| \leq C \left(\mu_\alpha^{\frac{n-4}{2}} r_\alpha^{4-n-j} + \|u_\infty\|_{L^\infty} \right) \tag{III.44}$$

in M, for $j = 0, 1, 2, 3$ and all α, where r_α and μ_α are defined as

$$\begin{cases} r_\alpha(x) = \min_{i=1,\dots K} d_g(x_{i,\alpha}, x) \\ \mu_\alpha := \max_{i=1,\dots K} \mu_{i,\alpha}. \end{cases} \tag{III.45}$$

We start by extending Corollary 1.5 to the higher derivatives of u_α.

Lemma 2.2. Under the assumptions of the Theorem 2.1, there exist a $C > 0$ such that up to a subsequence

$$r_a^{\frac{n-4}{2}+k} \left|\nabla^k u_\alpha\right| \leq C \quad \text{in } M, \text{ for all } \alpha \text{ and all } k = 1, 2, 3. \tag{III.46}$$

Proof. Let G_α be the Green's function of the operator $\Delta_g^2 + b_\alpha \Delta_g + c_\alpha$, Green's representation formula gives us, since u_α satisfies (III.43),

$$u_\alpha(x) = \int_M G_\alpha(x; y) u_\alpha(y)^{2^\#-1} dv_g(y) \tag{III.47}$$

for all $x \in M$. Applying the derivative we have

$$\nabla^k u_\alpha(x) = \int_M \nabla_x^k G_\alpha(x; y) u_\alpha(y)^{2^\#-1} dv_g(y). \tag{III.48}$$

Here the index x indicates that the gradient is taken with respect to this first variable. We then use the estimate of Proposition II.2.11, $\left|\nabla_y^k G(x; y)\right| \leq C d_g(x, y)^{4-n-k}$, and the symmetry of G according to its two variables, and we get

$$\left|\nabla^k u_\alpha(x)\right| \leq C \int_M d_g(x, y)^{4-n-k} u_\alpha(y)^{2^\#-1} dv_g(y). \tag{III.49}$$

Finally we use the estimate on u_α, $u_\alpha(y) \leq C \left(\min_{i=1,\dots K} d_g(x_{i,\alpha}, y)\right)^{-\frac{n-4}{2}}$, then

$$\begin{aligned} \left|\nabla^k u_\alpha(x)\right| &\leq C \max_{i=1,\dots K} \int_M d_g(x, y)^{4-n-k} d_g(x_{i,\alpha}, y)^{-\frac{n+4}{2}} dv_g(y) \\ &\leq C \max_{i=1,\dots K} \left(d_g(x, x_{i,\alpha})^{-\frac{n-4}{2}-k}\right) = C r_\alpha(x)^{-\frac{n-4}{2}-k} \end{aligned} \tag{III.50}$$

using Giraud's Lemma II.2.12, since $k = 1, 2, 3$. □

The estimate of Proposition 1.5 and Lemma 2.2 are scale invariant around the blow-up points. For instance, in the Euclidean setup with a single isolated bubble placed at $x = 0$, these results give us

$$\begin{cases} |x|^{\frac{n-4}{2}} |u(x)| \leq C \\ |x|^{\frac{n}{2}} |\Delta u(x)| \leq C \end{cases} \tag{III.51}$$

which are both invariant under the rescaling $u \rightsquigarrow \lambda^{(n-4)/2} u(\lambda x)$. The rescaling corresponds to the transformation R_0^λ, that also leaves invariant the limit equation $\Delta^2 u = u^{2^\#-1}$ and the norms $\|\cdot\|_{L^{2^\#}}$ and $\|-\Delta\, \cdot\|_{L^2}$.

As already mentioned, one of the only methods we have to study the fourth degree is to decompose it in a system of two equations. From now on we will use v_α defined as

$$v_\alpha := \Delta_g u_\alpha + \frac{b_\alpha}{2} u_\alpha, \tag{III.52}$$

then v_α must solve

$$\Delta_g v_\alpha + \frac{b_\alpha}{2} v_\alpha = \tilde{c}_\alpha u_\alpha + u_\alpha^{2^\#-1}, \tag{III.53}$$

where $\tilde{c}_\alpha := \frac{b_\alpha^2}{4} - c_\alpha > 0$ for $\alpha \gg 1$. We can then use well known second degree elliptic theory. First by the maximum principle on (III.53) we must have either $v_\alpha > 0$ on M or $v_\alpha \equiv 0$, for $\alpha \gg 1$. Furthermore, if we suppose $b_\alpha > 0$ it implies $v_\alpha > 0$. Finally we have the following Schauder estimate in the case that $u_\alpha,\ v_\alpha$ are sufficiently regular.

Proposition 2.3. With v_α defined as in (III.52), satisfying (III.53) and $v_\alpha \in C^{k,\alpha}(\Omega)$ for some $\alpha > 0$, then we get $u_\alpha \in C^{k+2,\alpha}(\Omega)$ where $\Omega \subset M$ and for all $U \subset\subset \Omega$, $v \in C^{k+2,\alpha}(U)$ with

$$\|v_\alpha\|_{C^{k+2}(U)} \leq C \left(\|u_\alpha\|_{C^{k,\alpha}(\Omega)} + \|v_\alpha\|_{L^1(\Omega)} \right). \tag{III.54}$$

Proposition 2.4. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, $(b_\alpha), (c_\alpha)$ converging sequences of real numbers such that $\frac{b_\alpha^4}{4} - c_\alpha \geq 0$ for all α, and (u_α) a bounded sequence of functions in H^2 that are positive solutions to the equation (III.43) such that $\|u_\alpha\|_{L^\infty} \to +\infty$.
Then there exists $C_1 > 0$ such that

$$v_\alpha \geq C_1 u_\alpha^{2^\#/2} \quad \text{in } M \text{ for all } \alpha. \tag{III.55}$$

Furthermore if one of the two conditions $u_\infty \not\equiv 0$ or $\frac{b^2}{4} - c > 0$ holds, where u_∞ is the weak limit in H^2 of the bounded sequence (u_α) and b, c are the respective limits of the $(b_\alpha), (c_\alpha)$, then there is $C_2 > 0$ such that

$$v_\alpha \geq C_2 u_\alpha \quad \text{in } M \text{ for all } \alpha. \tag{III.56}$$

Remark : We will need the fact that if $\Omega \subset M$ is open, and $u, v \in C^2(\Omega)$ are two positive functions. If x_0 is a point where $\frac{v}{u}$ reaches its maximum inside Ω, then since

$$\nabla \left(\frac{v}{u} \right) = \frac{u \nabla v - v \nabla u}{u^2} \quad \Rightarrow u(x_0) \nabla v(x_0) = v(x_0) \nabla u(x_0), \tag{III.57}$$

we obtain

$$\Delta_g \left(\frac{v}{u} \right)(x_0) = \frac{u(x_0) \Delta_g v(x_0) - v(x_0) \Delta_g u(x_0)}{u(x_0)^2}. \tag{III.58}$$

Finally, since x_0 reaches the maximum of v/u, $\Delta_g \left(\frac{v}{u} \right)(x_0) \geq 0$ and we have

$$\frac{\Delta_g v(x_0)}{v(x_0)} \geq \frac{\Delta_g u(x_0)}{u(x_0)}. \tag{III.59}$$

Proof. To compare $u_\alpha^{2^\#/2}$ and v_α, we will start by defining $x_\alpha \in M$ such that

$$\frac{u_\alpha(x_\alpha)^{2^\#/2}}{v_\alpha(x_\alpha)} = \max_{x \in M} \frac{u_\alpha^{2^\#/2}}{v_\alpha}. \tag{III.60}$$

Using the previous Remark, by (III.59) we have

$$\frac{\Delta_g\big(u_\alpha^{2^\#/2}\big)(x_\alpha)}{u_\alpha(x_\alpha)^{2^\#/2}} \geq \frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)}. \tag{III.61}$$

But $\Delta_g(u_\alpha^{2^\#/2}) = \frac{2^\#}{2}u_\alpha^{2^\#/2-1}\Delta_g u_\alpha - \frac{2^\#}{2}\left(\frac{2^\#}{2}-1\right)u_\alpha^{2^\#/2-2}\,|\nabla u_\alpha|^2$, so that

$$\frac{2^\#}{2}\frac{\Delta_g u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} = \frac{2^\#}{2}\frac{u_\alpha^{2^\#/2-1}\Delta_g u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)^{2^\#/2}} \geq \frac{\Delta_g\big(u_\alpha^{2^\#/2}\big)(x_\alpha)}{u_\alpha(x_\alpha)^{2^\#/2}} \geq \frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)}. \tag{III.62}$$

By the definition of $v_\alpha = \Delta_g u_\alpha + \frac{b_\alpha}{2}u_\alpha$, we know

$$\begin{aligned}\frac{2^\#}{2}\frac{v_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} &= \frac{2^\#}{2}\frac{\Delta_g u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} + \frac{2^\#}{2}\frac{b_\alpha}{2}\\ &\geq \frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)} + \frac{2^\#}{4}b_\alpha\\ &\geq \tilde{c}_\alpha\frac{u_\alpha(x_\alpha)}{v_\alpha(x_\alpha)} + \frac{u_\alpha(x_\alpha)^{2^\#-1}}{v_\alpha(x_\alpha)} + \left(\frac{2^\#}{2}-1\right)\frac{b_\alpha}{2}\end{aligned} \tag{III.63}$$

since v_α satisfies $\Delta_g v_\alpha + \frac{b_\alpha}{2}v_\alpha = \tilde{c}_\alpha u_\alpha + u_\alpha^{2^\#-1}$, with $\tilde{c}_\alpha = \frac{b_\alpha^2}{4} - c_\alpha$, and $b_\alpha > 0, \tilde{c}_\alpha \geq 0$ by assumption. We obtain

$$v_\alpha(x_\alpha) \geq \sqrt{\frac{2}{2^\#}}u_\alpha(x_\alpha)^{2^\#/2} \quad \forall\ \alpha. \tag{III.64}$$

Since by definition x_α reaches the maximum of $\frac{u_\alpha^{2^\#/2}}{v_\alpha}$ on M, we obtain indeed the first estimate.

Assume now that either $u_\infty \not\equiv 0$ or $\frac{b^2}{4} - c > 0$. Let us compare u_α and v_α, we set x_α such as

$$\frac{u_\alpha(x_\alpha)}{v_\alpha(x_\alpha)} = \max_{x\in M}\frac{u_\alpha(x)}{v_\alpha(x)}, \tag{III.65}$$

we also get by (III.59)

$$\frac{\Delta_g u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \geq \frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)}. \tag{III.66}$$

Using the expressions (III.52) and (III.53) that v_α satisfies,

$$\begin{aligned}\frac{v_\alpha - \frac{b_\alpha}{2}}{u_\alpha}(x_\alpha) &= \frac{\Delta_g u_\alpha}{u_\alpha}(x_\alpha)\\ &\geq \frac{\Delta_g v_\alpha}{v_\alpha}(x_\alpha) = \frac{\Delta_g v_\alpha + \frac{b_\alpha}{2}v_\alpha}{v_a}(x_\alpha) - \frac{b_\alpha}{2}\\ &\geq \tilde{c}_\alpha\frac{u_\alpha}{v_\alpha}(x_\alpha) + \frac{u_\alpha^{2^\#-1}}{v_\alpha}(x_\alpha) - \frac{b_\alpha}{2}\\ \Rightarrow \frac{v_\alpha(x_\alpha)^2}{u_\alpha(x_\alpha)^2} &\geq \tilde{c}_\alpha + u_\alpha(x_\alpha)^{2^\#-2}.\end{aligned} \tag{III.67}$$

Now if $\frac{b^2}{4} - c > 0$, there is $\delta > 0$ such that $\tilde{c}_\alpha \geq \delta \ \forall \ \alpha$, thus $\tilde{c}_\alpha + u_\alpha(x_\alpha)^{2^\#-2} \geq C$ independently of α. Otherwise, if $u_\infty \not\equiv 0$, we write G_α the Green's function of the operator $\Delta_g^2 + b_\alpha \Delta_g + c_\alpha$ and we take $\delta' > 0$,

$$u_\alpha(x_\alpha) = \int_M G_\alpha(x_\alpha; \ \cdot \ ) u_\alpha^{2^\#-1} \ dv_g \geq \int_{M \setminus \cup_i B_{x_{i,\alpha}}(\delta')} G_\alpha(x_\alpha; \ \cdot \ ) u_\alpha^{2^\#-1} \ dv_g. \tag{III.68}$$

We know that $u_\alpha \to u_\infty$ punctually in all compact subset $K \subset M \setminus \cup_i B_{x_{i,\alpha}}(\delta')$, so that

$$u_\alpha(x_\alpha) \geq \int_K G_\alpha(x_\alpha; \ \cdot \ ) u_\infty^{2^\#-1} dv_g \geq \delta \tag{III.69}$$

for some $\delta > 0$, since G_α is a positive function.

In either case we obtain

$$\frac{v_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \geq \sqrt{\tilde{c}_\alpha + u_\alpha(x_\alpha)^{2^\#-2}} \geq C \tag{III.70}$$

independently of α. □

The proof of Theorem 2.1 goes through the following three Lemmas. Let us start by proving ε-sharp estimates on v_α. We define, for $\delta > 0$,

$$\gamma_\alpha(\delta) := \max_{M \setminus \cup_i B_{x_{i,\alpha}}(\delta)} v_\alpha. \tag{III.71}$$

By the Proposition 1.2, we already know that $R_{\mu_{i,\alpha}}^{x_{i,\alpha}} u_\alpha \to \mathrm{b}$ in $C^4_{loc}(\mathbb{R}^n \setminus \mathcal{S})$, and thus at a distance $\mu_{i,\alpha}$ of the concentration center $u_\alpha \sim \mu_{i,\alpha}^{-\frac{n-4}{2}}$. The estimate of Corollary 1.5 is then useful up until this radius, after which it underestimates the rate of decay of u_α. At great distances of the center, we have a different behaviour and the following two estimates allow us to interpolate between those two scales, starting with v_α.

Lemma 2.5. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, $b, c > 0$ positive real numbers such that $\frac{b^2}{4} - c > 0$. Taking $(b_\alpha), (c_\alpha)$ sequences of real numbers converging to b and c respectively, and (u_α) a bounded sequence of functions in H^2 that are positive non-trivial solutions to the equation (III.43) that satisfies (III.2). Let $\varepsilon_0 > 0$ small enough, then for all $0 < \varepsilon < \varepsilon_0$, there exists $R_\varepsilon > 0, \delta_\varepsilon > 0$ and $C_\varepsilon > 0$ such that up to a subsequence

$$v_\alpha \leq C_\varepsilon \left(\mu_\alpha^{\frac{n-4}{2} - (n-2)\varepsilon} r_\alpha^{(2-n)(1-\varepsilon)} + \gamma_\alpha(\delta_\varepsilon) r_\alpha^{(2-n)\varepsilon} \right) \tag{III.72}$$

for all α in $M \setminus \cup_i B_{x_{i,\alpha}}(R_\varepsilon \mu_{i,\alpha})$

Proof. We start by noticing that in the estimate we want to obtain, we observe the presence of the expression r_α^{2-n}, but we recall that it is the order of magnitude of the Green's function for a Laplacian. We then write $\tilde{G}_1$ the Green function of the coercive operator $\Delta_g + 1$ for which $\tilde{G}_1(x_{i,\alpha}; x) \leq C r_\alpha(x)^{-(n-2)}$. This leads us to define a function having the required form, fixing $\varepsilon > 0, \delta > 0$,

$$\psi_{\alpha,\varepsilon}(x) := \mu_\alpha^{\frac{n-4}{2} - (n-2)\varepsilon} \sum_{i=1}^K \tilde{G}_1(x_{i,\alpha}; x)^{1-\varepsilon} + \gamma_\alpha(\delta) \sum_{i=1}^K \tilde{G}_1(x_{i,\alpha}; x)^\varepsilon. \tag{III.73}$$

We also fix $R > 0$, and we define

$$\Omega_{\alpha,R} := \bigcup_{i=1}^{K} B_{x_{i,\alpha}}(R\mu_{i,\alpha}), \tag{III.74}$$

as in the proof of the previous Proposition, we take $x_\alpha \in M \setminus \Omega_{\alpha,R}$ such that

$$\max_{M\setminus\Omega_{\alpha,R}} \frac{v_\alpha}{\psi_{\alpha,\varepsilon}} = \frac{v_\alpha(x_\alpha)}{\psi_{\alpha,\varepsilon}(x_\alpha)}. \tag{III.75}$$

Since $\psi_{\alpha,\varepsilon}$ is defined independently from u_α, only using the Green's function, we use the property of those function using the distance from their singularity : There exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$ we can find two constants $C_0(\varepsilon), C_1(\varepsilon) > 0$ such that

$$r_\alpha^2 \frac{\Delta_g \psi_{\alpha,\varepsilon}}{\psi_{\alpha,\varepsilon}} \geq C_0 - C_1 r_\alpha^2. \tag{III.76}$$

Supposing by contradiction that x_α is the interior of $M \setminus \Omega_{\alpha,R}$ and that $r_\alpha(x_\alpha) < \delta$, then (III.59) gives

$$\frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)} \geq \frac{\Delta_g \psi_{\alpha,\varepsilon}(x_\alpha)}{\psi_{\alpha,\varepsilon}(x_\alpha)}. \tag{III.77}$$

Notice that since we evaluate this expression in $M \setminus \Omega_{\alpha,R}$, we don't have to deal with the singularity we would get applying a Laplacian to the Green's function $\tilde{G}_1$ at its singularity. Here $\Delta_g \psi_{\alpha,\varepsilon}$ is well-behaved.

Since $\Delta_g v_\alpha(x_\alpha) = u_\alpha(x_\alpha)^{2^\#-1} + \tilde{c}_\alpha u_\alpha(x_\alpha) - b_\alpha v_\alpha$,

$$r_\alpha(x_\alpha)^2 \frac{\Delta_g v_\alpha(x_\alpha)}{v_\alpha(x_\alpha)} \leq r_\alpha(x_\alpha)^2 \frac{u_\alpha(x_\alpha)^{2^\#-1}}{v_\alpha(x_\alpha)} + r_\alpha(x_\alpha)^2 \tilde{c}_\alpha \frac{u_\alpha(x_\alpha)}{v_\alpha(x_\alpha)},$$

using the two estimates on v_α of Proposition 2.4, since $\frac{b^2}{4} - c > 0$

$$\leq C r_\alpha(x_\alpha)^2 u_\alpha(x_\alpha)^{\frac{2^\#-2}{2}} + C r_\alpha(x_\alpha)^2,$$

$C > 0$ not depending on α since $\tilde{c}_\alpha$ must be bounded for $\alpha \to \infty$.

By the estimate of Proposition 1.3, we know

$$\begin{aligned} \left(r_\alpha(x_\alpha)^2 u_\alpha(x_\alpha)^{\frac{2^\#-2}{2}} \right)^2 &= r_\alpha(x_\alpha)^4 u_\alpha(x_\alpha)^{2^\#-2} \\ &\leq C \sum_{i=1}^{K} r_\alpha(x_a)^4 B_\alpha^i(x_\alpha)^{2^\#-2} + C r_\alpha(x_\alpha)^4 + o(1) \leq C\delta^4 + \varepsilon(R). \end{aligned} \tag{III.78}$$

This comes from the assumption that $r_\alpha(x_\alpha) \leq \delta$, and that $d_g(x_{i,\alpha}, \cdot)^{\frac{n-4}{2}} B_\alpha^i$ is uniformly bounded, with the bound $\varepsilon(R) \to 0$ as $R \to \infty$, i.e. making the domain $M \setminus \Omega_{\alpha,R}$ smaller and smaller. Putting (III.76)-(III.78) together, we should have

$$C_0 \leq C_1 \delta^2 + C\delta^4 + \varepsilon(R) + C\delta^2, \tag{III.79}$$

and that is a contradiction taking $R \gg 1$ big enough and $\delta \ll 1$ small enough.

Thus, either $x_\alpha \in \partial(M \setminus \Omega_{\alpha,R})$ or $r_\alpha(x_\alpha) \geq \delta$, and the maximum of v_α compared to $\psi_{\alpha,\varepsilon}$ must happen further than δ from the singularities $x_{i,\alpha}$, and x_α is in the interior of $M \setminus \Omega_{\alpha,R}$. Recalling the definition of $\gamma_\alpha(\delta)$ (III.71), we then have

$$\frac{v_\alpha(x_\alpha)}{\psi_{\alpha,\varepsilon}(x_\alpha)} \leq \frac{v_\alpha(x_\alpha)}{\gamma_\alpha(\delta)} \left(\sum_{i=1}^K \tilde{G}_1(x_{i,\alpha}; x_\alpha)^\varepsilon \right)^{-1} \leq C. \tag{III.80}$$

On the other hand, if $x_\alpha \in \partial\left(M \setminus \Omega_{\alpha,R}\right)$, then fixing i and writing $x_\alpha = \exp_{x_{i,\alpha}}(\mu_{i,\alpha} z_\alpha)$ for some z_α,

$$v_\alpha(x_\alpha) = \mu_{i,\alpha}^{-\frac{n}{2}} \left(\Delta_{g_\alpha} \left(R^{\mu_{i,\alpha}}_{x_{i,\alpha}} u_\alpha \right) + b_\alpha \frac{\mu_{i,\alpha}^2}{2} R^{\mu_{i,\alpha}}_{x_{i,\alpha}} u_\alpha \right)(z_\alpha) \tag{III.81}$$

where $g_\alpha(x) = \left(\exp^*_{x_{i,\alpha}} g \right)(\mu_{i,\alpha} x)$. Then by Proposition 1.2, we have by the expression of the bubble and properties of the Green's function

$$\frac{v_\alpha(x_\alpha)}{\psi_{\alpha,\varepsilon}(x_\alpha)} \leq C \frac{\mu_{i,\alpha}^{-\frac{n}{2}} \mu_{i,\alpha}^{(n-2)(1-\varepsilon)}}{\mu_{i,\alpha}^{\frac{n-4}{2}-(n-2)\varepsilon}} = C_\varepsilon. \tag{III.82}$$

In either case we obtained $\frac{v_\alpha(x_\alpha)}{\psi_{\alpha,\varepsilon}(x_\alpha)} \leq C_\varepsilon$, so using $\tilde{G}_1(x_{i,\alpha}; x) \leq C r_\alpha(x)^{-(n-2)}$ we can conclude,

$$v_\alpha(x) \left(\mu_\alpha^{\frac{n-4}{2}-(n-2)\varepsilon} r_\alpha^{(2-n)(1-\varepsilon)} + \gamma_\alpha(\delta_\varepsilon) r_\alpha^{(2-n)\varepsilon} \right)^{-1} \leq \frac{v_\alpha(x_\alpha)}{\psi_{\alpha,\varepsilon}(x_\alpha)} \leq C_\varepsilon \tag{III.83}$$

in $M \setminus \Omega_{\alpha,R_\varepsilon}$. □

This allows us to get similar ε-sharp estimates on u_α, in $M \setminus \cup_i B_{x_{i,\alpha}}(R_\varepsilon \mu_{i,\alpha})$.

Lemma 2.6. Under the assumptions of Lemma 2.5, we have for $0 < \varepsilon < \varepsilon_0$, a $R_\varepsilon > 0, \delta_\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$u_\alpha \leq C_\varepsilon \left(\mu_\alpha^{\frac{n-4}{2}-(n-2)\varepsilon} r_\alpha^{4-n+(n-2)\varepsilon} + \gamma_\alpha(\delta_\varepsilon) \right) \tag{III.84}$$

in $M \setminus \cup_i B_{x_{i,\alpha}}(R_\varepsilon \mu_{i,\alpha})$ for all α.

Proof. In a similar way than before, we will write $\tilde{G}_b$ the Green's function of $\Delta_g + \frac{b}{2}$. We choose for all α $x_\alpha^0 \in M \setminus \Omega_{\alpha,R}$, where R will be chosen later. By definition of a Green function,

$$\begin{aligned} u_\alpha(x_\alpha^0) &= \int_M \tilde{G}_b(x_\alpha^0; x) \left((\Delta_g + \frac{b}{2}) u_\alpha \right)(x) dv_g(x) \\ &\leq \int_M \tilde{G}_b(x_\alpha^0; x) v_\alpha(x) dv_g(x) \\ &\leq \int_{M \setminus \Omega_{\alpha,R_\varepsilon}} \tilde{G}_b(x_\alpha^0; x) v_\alpha(x) dv_g(x) + \sum_{i=1}^K \int_{B_{x_{i,\alpha}}(R_\varepsilon \mu_{i,\alpha})} \tilde{G}_b(x_\alpha^0; x) v_\alpha(x) dv_g(x) \end{aligned} \tag{III.85}$$

for $\alpha \gg 1$, where R_ε is the radius we obtained for the estimates on v_α.

First of all we know that $\tilde{G}_b(x_\alpha^0; x) \leq C d_g(x_\alpha^0, x)^{2-n}$, and taking $R \geq 2R_\varepsilon$, we know that $d_g(x_\alpha^0, x) \geq \frac{1}{2} d_g(x_\alpha^0, x_{i,\alpha})$ inside $B_{x_{i,\alpha}}(R_\varepsilon \mu_{i,\alpha})$, so

$$\begin{aligned}\int_{B_{x_{i,\alpha}}(\mu_{i,\alpha})} \tilde{G}_b(x_\alpha^0; x) v_\alpha(x) dv_g(x) &\leq C d_g(x_\alpha^0, x_{i,\alpha})^{2-n} \int_{B_{x_{i,\alpha}}(\mu_{i,\alpha})} v_\alpha dv_g \\ &\leq C d_g(x_\alpha^0, x_{i,\alpha})^{2-n} \mathrm{Vol}_g(B_{x_{i,\alpha}}(R_\varepsilon \mu_{i,\alpha}))^{1/2} \|v_\alpha\|_{L^2} \\ &\leq C d_g(x_\alpha^0, x_{i,\alpha})^{2-n} \mu_{i,\alpha}^{n/2}\end{aligned} \tag{III.86}$$

since v_α is bounded in H^2 and $\mathrm{Vol}_g(B_{x_{i,\alpha}}(R_\varepsilon \mu_{i,\alpha})) = C\mu_{i,\alpha}^n$.

On the other integral, we use the estimate of v_α in $M \setminus \Omega_{\alpha, R_\varepsilon}$,

$$v_\alpha \leq C_\varepsilon \left(\mu_\alpha^{\frac{n-4}{2} - (n-2)\varepsilon} r_\alpha^{(2-n)(1-\varepsilon)} + \gamma_\alpha(\delta_\varepsilon) r_\alpha^{(2-n)\varepsilon} \right), \tag{III.87}$$

and we compute

$$\begin{aligned}\int_{M \setminus \Omega_{\alpha, R_\varepsilon}} \tilde{G}_b(x_\alpha^0; x) r_\alpha(x)^{(2-n)(1-\varepsilon)} dv_g(x) &\leq C \sum_{i=1}^K \int_M d_g(x_\alpha^0, x)^{2-n} d_g(x_{i,\alpha}, x)^{-n+2+\varepsilon(n-2)} dv_g \\ &\leq C \sum_{i=1}^K d_g(x_\alpha^0, x_{i,\alpha})^{-n+4+\varepsilon(n-2)}\end{aligned} \tag{III.88}$$

by Giraud's Lemma, if $\varepsilon < \frac{n-4}{n-2}$. We also have, using Giraud's Lemma once again

$$\begin{aligned}\int_{M \setminus \Omega_{\alpha, R_\varepsilon}} \tilde{G}_b(x_\alpha^0; x) r_\alpha(x)^{(2-n)\varepsilon} dv_g &\leq C \sum_{i=1}^K \int_M d_g(x_{i,\alpha}, x)^{2-n} d_g(x_\alpha^0, x)^{(2-n)\varepsilon} dv_g \\ &\leq C \sum_{i=1}^K d_g(x_a^0; x_{i,\alpha})^{2+(2-n)\varepsilon}\end{aligned} \tag{III.89}$$

for $\varepsilon < \frac{2}{n-2}$, so that for $x_\alpha^0 \in M \setminus \cup_i B_{x_{i,\alpha}}(R_\varepsilon \mu_{i,\alpha})$, $d_g(x_a^0; x_{i,\alpha})^{2+(2-n)\varepsilon} \leq C$.

Coming back to (III.85), we get

$$u_\alpha(x_\alpha^0) \leq C\mu_\alpha^{\frac{n-4}{2} - (n-2)\varepsilon} \sum_{i=1}^K d_g(x_\alpha^0, x_{i,\alpha})^{-n+4+\varepsilon(n-2)} + C\gamma_\alpha(\delta_\varepsilon) + C\mu_{i,\alpha}^{n/2} \sum_{i=1}^K d_g(x_\alpha^0, x_{i,\alpha})^{2-n}, \tag{III.90}$$

but we realise

$$\begin{aligned}&\mu_\alpha^{\frac{n-4}{2} - (n-2)\varepsilon} d_g(x_\alpha^0, x_{i,\alpha})^{-n+4+\varepsilon(n-2)} + \mu_{i,\alpha}^{n/2} d_g(x_\alpha^0, x_{i,\alpha})^{2-n} \\ &\quad = \mu_{i,\alpha}^{\frac{n-4}{2} - (n-2)\varepsilon} d_g(x_\alpha^0, x_{i,\alpha})^{-n+4+\varepsilon(n-2)} \left(1 + \mu_{i,\alpha}^{n/2} \mu_\alpha^{-\frac{n-4}{2} + (n-2)\varepsilon} d_g(x_\alpha^0, x_{i,\alpha})^{-2-\varepsilon(n-2)} \right)\end{aligned} \tag{III.91}$$

with

$$\mu_{i,\alpha}^{n/2} \mu_\alpha^{-\frac{n-4}{2} + \varepsilon(n-2)} d_g(x_\alpha^0, x_{i,\alpha})^{-2-(n-2)\varepsilon} \leq C \left(\frac{\mu_{i,\alpha}}{\mu_\alpha} \right)^{\frac{n-4}{2} - (n-2)\varepsilon} \leq C \tag{III.92}$$

since $d_g(x^0_\alpha, x_{i,\alpha}) \geq R_\varepsilon \mu_{i,\alpha}$ and $\mu_\alpha = \max_i \mu_{i,\alpha}$. We can finally conclude, whatever the $x \in M \backslash \Omega_{\alpha,R_\varepsilon}$, and recalling $d_g(x, x_{i,\alpha}) \geq r_\alpha(x)$,

$$u_\alpha(x) \leq C \left(\mu_{i,\alpha}^{\frac{n-4}{2}-\varepsilon(n-2)} r_\alpha(x)^{-n+4+\varepsilon(n-2)} + \gamma_\alpha(\delta_\varepsilon) \right). \tag{III.93}$$

□

We will use these estimates to get inequalities involving the higher order derivatives of u_α, that will improve Lemma 2.2.

Lemma 2.7. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, $b, c > 0$ positive real numbers such that $\frac{b^2}{4} - c > 0$. Taking $(b_\alpha), (c_\alpha)$ sequences of real numbers converging to b and c respectively, and (u_α) a bounded sequence of functions in H^2 that are positive solutions to the equation (III.43) that explode. Then there exist $C > 0$, $R > 0$ and $\delta > 0$ such that, up to a subsequence

$$\left|\nabla^j u_\alpha\right| \leq C \left(\mu_\alpha^{\frac{n-4}{2}} r_\alpha^{4-n-j} + \gamma_\alpha(\delta)^{2^\#-1} \right) \tag{III.94}$$

in $M \setminus \Omega_{\alpha,R}$ for $j = 0, 1, 2, 3$.

Proof. We will use the previously obtained ε-sharp estimate in Green's representation formula to allow us drop the ε exponent. We write G_α the Green's function of the operator $\Delta_g^2 + b_\alpha \Delta_g + c_\alpha$, and we know that for ε small enough we have $R_\varepsilon > 0, \delta_\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$u_\alpha \leq C_\varepsilon \left(\mu_\alpha^{\frac{n-4}{2}-(n-2)\varepsilon} r_\alpha^{4-n+(n-2)\varepsilon} + \gamma_\alpha(\delta_\varepsilon) \right) \quad \text{in } M \setminus \cup_i B_{x_{i,\alpha}}(R_\varepsilon \mu_{i,\alpha}). \tag{III.95}$$

We take for all α, $x^0_\alpha \in M \setminus \Omega_{\alpha,R}$ with $R = 2R_\varepsilon + 1$, since

$$u_\alpha(x) = \int_M G_\alpha(x; y) u_\alpha(y)^{2^\#-1} dv_g(y) \quad \forall\ \alpha,\ \forall\ x \in M, \tag{III.96}$$

and using that $\left|\nabla_y^j G_\alpha(x; y)\right| \leq C d_g(x, y)^{4-n-k}$, we will decompose our integral in several parts,

$$\begin{aligned} \left|\nabla^j u_\alpha(x^0_\alpha)\right| &\leq \int_M \left|\nabla_x^j G_\alpha(x^0_\alpha; x)\right| u_\alpha^{2^\#-1} dv_g(x) \\ &\leq C \int_M d_g(x^0_\alpha, x)^{4-n-j} u_\alpha(x)^{2^\#-1} dv_g(x) \\ &\leq C \Big(\int_{M \setminus \Omega_{\alpha,R_\varepsilon}} d_g(x^0_\alpha, x)^{4-n-j} u_\alpha(x)^{2^\#-1} dv_g(x) \\ &\qquad + \int_{\Omega_{\alpha,R_\varepsilon}} d_g(x^0_\alpha, x)^{4-n-j} u_\alpha(x)^{2^\#-1} dv_g(x) \Big). \end{aligned} \tag{III.97}$$

We write $k_n := (2^\# - 1)(n-2)$ for simplification, then using the estimate

$$\int_{M \setminus \Omega_{\alpha,R_\varepsilon}} d_g(x^0_\alpha, x)^{4-n-j} u_\alpha(x)^{2^\#-1} dv_g(x) \leq C \mu_\alpha^{\frac{n+4}{2}-k_n\varepsilon} \sum_{i=1}^K \mathcal{A}^i_\alpha + C \gamma_\alpha(\delta_\varepsilon)^{2^\#-1}, \tag{III.98}$$

where $\mathcal{A}^i_\alpha := \int_{M\setminus B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})} d_g(x^0_\alpha,x)^{4-n-j} d_g(x_{i,\alpha},x)^{-(4+n)+k_n\varepsilon} dv_g(x)$ for $i=1,\dots K$. We further split this integral in two, between $K_{i,\alpha} := \{x\in M \quad \text{s.t. } d_g(x^0_\alpha,x)\le \frac{1}{2} d_g(x_{i,\alpha},x^0_\alpha)\}$ and $K^c_{i,\alpha}$ the complementary set,

$$\begin{aligned}\mathcal{A}^i_\alpha \le &\int_{K_{i,\alpha}\setminus B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})} d_g(x^0_\alpha,x)^{4-n-j} d_g(x_{i,\alpha},x)^{-(4+n)+k_n\varepsilon} dv_g(x)\\ &+\int_{K^c_{i,\alpha}\setminus B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})} d_g(x^0_\alpha,x)^{4-n-j} d_g(x_{i,\alpha},x)^{-(4+n)+k_n\varepsilon} dv_g(x) =: \mathcal{A}^i_{\alpha 1}+\mathcal{A}^i_{\alpha 2}. \end{aligned} \tag{III.99}$$

Choosing ε small enough so that $4-k_n\varepsilon>0$, we have outside $B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})$ that $d_g(x_{i,\alpha},x)^{-4+k_n\varepsilon}\le C\mu_{i,\alpha}^{-4+k_n\varepsilon}$ so

$$\begin{aligned}\mathcal{A}^i_{\alpha 1} &\le C\mu_{i,\alpha}^{-4+k_n\varepsilon}\int_{K_{i,\alpha}\setminus B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})} d_g(x^0_\alpha,x)^{4-n-j-\theta} d_g(x_{i,\alpha},x)^{-n+\theta}\left(\frac{d_g(x^0_\alpha,x)}{d_g(x_{i,\alpha},x)}\right)^\theta dv_g(x)\\ &\le C\mu_{i,\alpha}^{-4+k_n\varepsilon} d_g(x^0_\alpha,x_{i,\alpha})^{4-n-j}\end{aligned} \tag{III.100}$$

using Giraud's Lemma since $j=0,1,2,3$ and choosing $\theta>0$ small enough. On the other hand in $K^c_{i,\alpha}$ we have $d_g(x^0_\alpha,x)>\frac{1}{2}d_g(x_{i,\alpha},x^0_\alpha)$ so that

$$\begin{aligned}\mathcal{A}^i_{\alpha 2} &\le C d_g(x_{i,\alpha},x^0_\alpha)^{4-n-j}\int_{K^c_{i,\alpha}\setminus B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})} d_g(x_{i,\alpha},x)^{-(n+4)+k_n\varepsilon} dv_g(x)\\ &\le C\mu_{i,\alpha}^{-4+k_n\varepsilon} d_g(x_{i,\alpha},x^0_\alpha)^{4-n-j}\end{aligned} \tag{III.101}$$

using here that $d_g(x_{i,\alpha},x)^{-n}$ is integrable on $M\setminus B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})$ and $d_g(x_{i,\alpha},x)^{-4+k_n\varepsilon}\le C\mu_{i,\alpha}^{-4+k_n\varepsilon}$ as before.

At last, since we took $x^0_\alpha\in M\setminus\Omega_{\alpha,R}$ with $R=2R_\varepsilon+1$, we know $d_g(x^0_\alpha,x)>\frac{1}{2}d_g(x_{i,\alpha},x^0_\alpha)$ for all $x\in B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})$. Thus using Hölder's inequality

$$\begin{aligned}&\int_{B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})} d_g(x^0_\alpha,x)^{4-n-j} u_\alpha(x)^{2^\#-1} dv_g(x)\\ &\qquad\le C d_g(x^0_\alpha,x_{i,\alpha})^{4-n-j}\int_{B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})} u_\alpha(x)^{2^\#-1} dv_g(x)\\ &\qquad\le C d_g(x^0_\alpha,x_{i,\alpha})^{4-n-j}\mathrm{Vol}_g(B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha}))^{1-\frac{2^\#-1}{2^\#}}\|u_\alpha\|^{2^\#-1}_{L^{2^\#}}\\ &\qquad\le C\mu_{i,\alpha}^{\frac{n-4}{2}} r_\alpha(x^0_\alpha)^{4-n-j}.\end{aligned} \tag{III.102}$$

Putting everything together we get

$$\begin{aligned}\left|\nabla^j u_\alpha(x^0_\alpha)\right| &\le C\mu_\alpha^{\frac{n+4}{2}-k_n\varepsilon}\sum_i \mu_{i,\alpha}^{-4+k_n\varepsilon} d_g(x^0_\alpha,x_{i,\alpha})^{4-n-j}\\ &\qquad + C\gamma_\alpha(\delta_\varepsilon)^{2^\#-1} + C\sum_i \mu_{i,\alpha}^{\frac{n-4}{2}} r_\alpha(x^0_\alpha)^{4-n-j}\\ &\le C\left(\mu_\alpha^{\frac{n-4}{2}} r_\alpha(x^0_\alpha)^{4-n-j}+\gamma_\alpha(\delta_\varepsilon)^{2^\#-1}\right)\end{aligned} \tag{III.103}$$

since we took ε such that $4 - k_n\varepsilon > 0$, and $\mu_\alpha \geq \mu_{i,\alpha}$ for all i. This holds for all $x_\alpha^0 \in M \setminus \Omega_{\alpha,R}$, if we take $R = 2R_\varepsilon + 1$ and $\delta = \delta_\varepsilon$ for some ε small enough we have the conclusion. □

We now have everything we need to prove our main result, that there is $C > 0$ such that

$$\left|\nabla^j u_\alpha\right| \leq C\left(\mu_\alpha^{\frac{n-4}{2}} r_\alpha^{4-n-j} + \|u_\infty\|_{L^\infty}\right). \tag{III.104}$$

The first thing we will need to do is to prove that we can replace $\gamma_\alpha(\delta)^{2^\#-1}$ by $\|u_\infty\|_{L^\infty}$, and then we will show that the estimates are global in M and not only outside $\Omega_{\alpha,R}$.

Proof (of Theorem 2.1). If $\|u_\infty\|_{L^\infty} \not\equiv 0$ there is nothing more to do, we can replace $\gamma_\alpha(\delta)$ by it since $\gamma_\alpha(\delta) \leq C$ for all α, by its definition and the Lemma 2.2 : v_α is expressed as a sum of u_α and $\Delta_g u_\alpha$, and both of those are bounded, and r_α also has a uniform minimum in $M \setminus \Omega_{\alpha,R}$. By changing the value of the constant C in front of the estimate it is over. The main thing to prove is then to show that we can drop the term $\gamma_\alpha(\delta)$ when u_∞ is zero.

We suppose $u_\infty \equiv 0$, and we define $\Omega_\alpha(\delta) := \cup_{i=1}^K B_{x_{i,\alpha}}(\delta)$. The estimates of Lemma 2.7 give us

$$\max_{M\setminus\Omega_\alpha(\delta/2)} u_\alpha \leq C\mu_\alpha^{\frac{n-4}{2}} + C\gamma_\alpha(\delta)^{2^\#-1}. \tag{III.105}$$

But since v_α is solution to the elliptic equation

$$\Delta_g v_\alpha + \frac{b_\alpha}{2} v_\alpha = \tilde{c}_\alpha u_\alpha + u_\alpha^{2^\#-1}, \tag{III.106}$$

and that u_α converges to u_∞ in $C^4_{loc}(M \setminus \Omega_\alpha(\delta))$, and thus $v_\alpha \in C^2_{loc}(M \setminus \Omega_\alpha(\delta))$, we know by the Schauder estimates that

$$\max_{M\setminus\Omega_\alpha(\delta)} v_\alpha \leq C \max_{M\setminus\Omega_\alpha(\delta/2)} u_\alpha + C\left\|v_\alpha\right\|_{L^1}. \tag{III.107}$$

Let us write $\tilde{G}_b$ the Green's function of $\Delta_g + b$, by its definition for $\alpha \gg 1$ we have $v_\alpha \leq (\Delta_g + b)\, u_\alpha$, and we know that there is Λ such that $\tilde{G}_b \geq \Lambda$ in M, thus

$$\Lambda\left\|v_\alpha\right\|_{L^1} \leq \int_M \tilde{G}_b(x;y) v_\alpha(y) dv_g(y) \leq u_\alpha(x) \tag{III.108}$$

for all x, the last inequality coming from the representation formula.

We then get, using (III.107)

$$\gamma_\alpha(\delta) = \max_{M\setminus\Omega_\alpha(\delta)} v_\alpha \leq C \max_{M\setminus\Omega_\alpha(\delta/2)} u_\alpha, \tag{III.109}$$

but since we assumed $u_\infty = 0$, by the estimates of Proposition 1.3, $u_\alpha \to 0$ in $L^\infty(M \setminus \Omega_\alpha(\delta/2))$ for $\alpha \to \infty$, so that $\gamma_\alpha(\delta) \to 0$ too. To conclude we just need to show that this convergence happens at least at a rate of $\mu_\alpha^{\frac{n-4}{2}}$. We know that by (III.105)

$$\begin{aligned} &\max_{M\setminus\Omega_\alpha(\delta/2)} u_\alpha \leq C\mu_\alpha^{\frac{n-4}{2}} + C\gamma_\alpha(\delta)^{2^\#-2} \max_{M\setminus\Omega_\alpha(\delta/2)} u_\alpha \\ \Rightarrow\ &\max_{M\setminus\Omega_\alpha(\delta/2)} u_\alpha \leq C\mu_\alpha^{\frac{n-4}{2}} \end{aligned} \tag{III.110}$$

and that concludes the first part, we have

$$\left|\nabla^j u_\alpha\right| \leq C\left(\mu_\alpha^{\frac{n-4}{2}} r_\alpha^{4-n-j} + \|u_\infty\|_{L^\infty}\right) \quad \text{in } M \setminus \cup_i B_{x_{i,\alpha}}(R\mu_{i,\alpha}). \tag{III.111}$$

To prove that these are in fact global estimates, we need to show that they hold in $\Omega_{\alpha,R}$. By Lemma 2.2 we have $r_\alpha^{\frac{n-4}{2}+j}\left|\nabla^j u_\alpha\right| \leq C$ in M, but inside of $\Omega_{\alpha,R}$ we know that $r_\alpha(x)^{-\frac{n-4}{2}-j} \leq C r_\alpha(x)^{4-n-j}$ and that concludes the proof. □

Chapter IV

Stability theorems

1 Low dimensions

We are now in possession of a description of blowing-up solutions for sequences of solutions satisfying

$$\Delta_g^2 u_\alpha + b_\alpha \Delta_g u_\alpha + c_\alpha u_\alpha = u_\alpha^{2^\#-1}. \tag{IV.1}$$

We are interested in perturbations around the Q-curvature's conformal transformation Equation, that is for an Einstein metric with $Ricc_g = \lambda g$,

$$\Delta_g^2 u + \frac{b_n \lambda}{n-1}\Delta_g u + \frac{c_n \lambda^2}{(n-1)^2} u = \frac{n-4}{2} Q_{\tilde{g}} u^{2^\#-1} \tag{IV.2}$$

with $b_n = \frac{n^2-2n-4}{2}$ and $c_n = \frac{n(n-4)(n^2-4)}{16}$. We know by Corollary II.3.13.ii that this equation has a solution $u \in C^4(M)$ with $u > 0$, we now wish to know whether it is stable. If (b_α), (c_α) are converging sequences of real numbers and (u_α) is a sequence of solutions in $C^4(M)$, can we say that $u_\alpha \to u_\infty$ in $C^4(M)$ for a $u_\infty \in C^4(M)$? Answering this question comes down to prove that there are no blowing-up bubbles for these sequences. To do that we need to understand the asymptotic behaviour of sequences of solutions. The purpose of this section will be to prove the following Theorem that holds for small dimensions, and then to get better conditions for the stability of the equation when $n = 5, 6$.

Theorem 1.1 (Strong convergence in low dimensions). Let (M, g) be a compact, locally conformally flat Riemannian manifold of dimension $n = 5, 6, 7$, and $b, c > 0$ positive real numbers such that $\frac{b^2}{4} - c > 0$. Let $(b_\alpha), (c_\alpha)$ sequences of real numbers converging respectively to b and c, and (u_α) a bounded sequence of functions in $H^2(M)$ that are positive non-trivial solutions to the equation (IV.1) such that $u_\alpha \rightharpoonup u_\infty$ weakly in $H^2(M)$. Then either $u_\infty \equiv 0$ or $u_\alpha \to u_\infty$ strongly in every $C^k(M)$.

The interaction radius. Before we can start to show some stability results, we will need to better understand the behaviour of our sequences of blowing-up solutions. Let us begin with the presentation of some premiliminary propositions giving the shape of the (u_α) when looked at a particular scale, this comes from Hebey, Robert, and Wen (2006).

Let (M, g) be a compact, locally conformally flat Riemannian manifold of dimension $n \geq 5$, with and $b, c > 0$ positive real numbers such that $\frac{b^2}{4} - c > 0$. Taking $(b_\alpha), (c_\alpha)$ sequences of real numbers

converging respectively to b and c, and (u_α) a bounded sequence of functions in $H^2(M)$ that are positive non-trivial solutions to the equation (IV.1) such that $u_\alpha \rightharpoonup u_\infty$ weakly in $H^2(M)$. We will start by reducing the problem and be interested in the dominant bubble, i.e. the bubble staying the most relevant as $\alpha \to \infty$. Without loss of generality, up to taking a subsequence we can assume

$$\mu_\alpha := \max_{i=1,\dots K} \mu_{i,\alpha} = \mu_{1,\alpha} \quad \forall\ \alpha, \tag{IV.3}$$

we then take $x_\alpha := x_{1,\alpha}$ the singularity point associated to μ_α. Writing $B_\alpha = B^1_\alpha$ the associated bubble, that is the lowest one, we can verify that evaluating B_α in points further from its center than $\sqrt{\mu_\alpha}$, this bubble tends locally to 0. Precisely, what we mean is that if $(\delta_\alpha)_\alpha$ is a sequence of real numbers $\delta_\alpha > 0$ such that $\frac{\delta_\alpha}{\sqrt{\mu_\alpha}} \to \infty$, and $(\tilde{x}_\alpha)$ is any sequence of points each in $M \setminus B_{x_\alpha}(\delta_\alpha)$, then $B_\alpha(\tilde{x}_\alpha) \to 0$. In other words

$$\lim_{\alpha\to\infty} \max_{M\setminus B_{x_\alpha}(R\sqrt{\mu_\alpha})} B_\alpha = \varepsilon_R \tag{IV.4}$$

with $\varepsilon_R \to 0$ as $R \to \infty$. This comes from the fact that

$$B_\alpha(x) = \eta(d_g(x,x_\alpha)) \left(\mu_\alpha + \left(\frac{d_g(x,x_\alpha)}{\sqrt{\mu_\alpha}} \right)^2 \frac{1}{\sqrt{\lambda_n}} \right)^{-\frac{n-4}{2}} \tag{IV.5}$$

with $\mu_\alpha \to 0$. It means that $\sqrt{\mu_\alpha}$ is exactly the distance around x_α in which all the bubble's explosion takes place.

In the same way we zoomed and then re-normalized around the concentration points in Proposition III.1.2, to obtain $R^{\mu_{i,\alpha}}_{x_{i,\alpha}} \left(\sum B^i_\alpha \right) \to \mathrm{b}$ in $L^\infty_{loc}(\mathbb{R}^n \setminus \mathcal{S}_i)$, here too we define the set of the images in the local chart around x_α of the blowing-up centers that stay relevant after zooming. We define

$$\begin{aligned} &\tilde{u}_\alpha(y) := u_\alpha(\exp_{x_\alpha}(\sqrt{\mu_\alpha} y)) \\ &\tilde{\mathcal{S}} := \{ \lim_{\alpha\to\infty} \frac{1}{\sqrt{\mu_\alpha}} \exp^{-1}_{x_\alpha}(x_{j,\alpha}) \in \mathbb{R}^n, j \in \tilde{I} \} \end{aligned} \tag{IV.6}$$

where $\tilde{I}$ is the set of indices $j = 1, \dots K$ for which $d_g(x_\alpha, x_{j,\alpha}) = O(\sqrt{\mu_\alpha})$. Hebey, Robert, and Wen (2006) then proved a result analogous to Proposition III.1.2.

Lemma 1.2. Under the above definitions, there is a $\tilde{u}_\infty \in C^4(\mathbb{R}^n \setminus \tilde{\mathcal{S}})$ such that up to a subsequence $\tilde{u}_\alpha \to \tilde{u}_\infty$ strongly in $C^3_{loc}(\mathbb{R}^n \setminus \tilde{\mathcal{S}})$. Moreover, $\tilde{u}_\infty$ is bi-harmonic, that is

$$\Delta^2 \tilde{u}_\infty = 0 \quad \text{in } \mathbb{R}^n \setminus \tilde{\mathcal{S}}, \tag{IV.7}$$

and $\tilde{u}_\infty \geq 0$, $-\Delta \tilde{u}_\infty \geq 0$.

Proof (A sketch of the proof). It comes from the fact that we have the decomposition

$$u_\alpha = u_\infty + \sum_{i=1}^K B^i_\alpha + R_\alpha \tag{IV.8}$$

with $\|R_\alpha\|_{H^2} \to 0$, and u_α satisfies (IV.1), so that $\tilde{u}_\alpha$ solves

$$\Delta^2_{g_\alpha} \tilde{u}_\alpha + b_\alpha \mu_\alpha \Delta_{g_\alpha} \tilde{u}_\alpha + c_\alpha \mu^2_\alpha \tilde{u}_\alpha = \mu^2_\alpha \tilde{u}^{2^\#-1}_\alpha \tag{IV.9}$$

with $g_\alpha(x) := (\exp^*_{x_\alpha} g)(\sqrt{\mu_\alpha} x)$ and $g_\alpha \to \xi$ in $C^2_{loc}(\mathbb{R}^n)$ the flat metric. By taking the limit, provided that $\tilde{u}_\alpha$ is bounded since u_α is, we have a weak limit $\tilde{u}_\infty$, satisfying the limiting equation

$$(-\Delta)^2 \tilde{u}_\infty = 0. \tag{IV.10}$$

This obviously can only hold pointwise where $\tilde{u}_\alpha$ does not blow up, so that justifies the introduction of $\tilde{\mathcal{S}}$. It remains to show the strong convergence and the positivity properties of $\tilde{u}_\infty$, we refer to Hebey, Robert, and Wen (2006) for that matter since it is based on results that will not be presented here. □

We can then describe $\tilde{u}_\infty$. It should be made of a component coming from the zoom into u_∞, and one for each bubble. Knowing then that $\tilde{u}_\infty$ is bi-harmonic on $\mathbb{R}^n \setminus \tilde{\mathcal{S}}$, and that $-\Delta \tilde{u}_\infty$ is harmonic on the same set, we can write

$$\tilde{u}_\infty(y) = \sum_{y_i \in \tilde{\mathcal{S}}} \frac{b_i}{|y - y_i|^{n-4}} + \sum_{y_i \in \tilde{\mathcal{S}}} \frac{a_i}{|y - y_i|^{n-2}} + \varphi(y) \tag{IV.11}$$

for all $y \in \mathbb{R}^n \setminus \tilde{\mathcal{S}}$, where φ is bi-harmonic on the whole $\mathbb{R}^n$.

Actually, since $-\Delta\varphi$ is harmonic on the whole space, but $-\Delta \tilde{u}_\infty \geq 0$, it must have at least one bound. But then by using the minimum and maximum principles, we get a generalized version of Liouville's Theorem telling us that $-\Delta\varphi = C_0$ a constant. In the same way, since $-\Delta\left(\frac{|x|^2}{2n}\right) = -1$, we get

$$-\Delta(\varphi + C_0 \frac{|x|^2}{2n}) = 0 \tag{IV.12}$$

and thus since $\tilde{u}_\infty \geq 0$, by its definition $|\varphi|$ is bounded below. This gives $\varphi = -C_0 \frac{|x|^2}{2n} + K_0$ by the same Liouville's Theorem argument. We then notice that φ is the only leading term for $|y| \to \infty$ in $\tilde{u}_\infty$ so that we must have $C_0 \geq 0$, and then likewise $-\Delta\varphi$ in the expression of $-\Delta \tilde{u}_\infty$ so $C_0 = 0$ and also $K_0 \geq 0$.

Moreover, since $0 \in \tilde{\mathcal{S}}$, around the point of focus x_α the dominant terms are

$$\frac{b_0}{|y|^{n-4}} + \frac{a_0}{|y|^{n-2}}, \tag{IV.13}$$

but knowing the shape of the bubble around 0, we must have $a_0 = 0$, and $b_0 \geq 0$. In the same way, all the other a_i, b_i must be positive constants. Writing $\tilde{\mathcal{S}} \setminus \{0\} = \{y_1, \dots y_L\}$ the remaining blowing-up centers, we then have

$$\tilde{u}_\alpha(y) \to \frac{A}{|y|^{n-4}} + \sum_{i=1}^{L} \left(\frac{b_i}{|y - y_i|^{n-4}} + \frac{a_i}{|y - y_i|^{n-2}} \right) + K_0 \quad \forall\, y \in \mathbb{R}^n \setminus \tilde{\mathcal{S}}. \tag{IV.14}$$

Finally, if $u_\infty \not\equiv 0$, using the regularity Theory and since P_g satisfies the pointwise comparison principle, we know $u_\infty \in C^4(M)$ and $u_\infty > 0$ on M. We write x_∞ the limit of x_α, and we define G_α the Green function of the operator $\Delta_g^2 + b_\alpha \Delta_g + c_\alpha$. Using Green's representation formula, we know

$$\begin{aligned} \tilde{u}_\alpha(y) &= \int_M G_\alpha(\exp_{x_\alpha}(\sqrt{\mu_\alpha} y); z) u_\alpha(z)^{2^\# - 1} dv_g(z) \\ &\geq \int_{M \setminus B_{x_0}(\delta)} G_\alpha(\exp_{x_\alpha}(\sqrt{\mu_\alpha} y); z) u_\alpha(z)^{2^\# - 1} dv_g(z) \end{aligned} \tag{IV.15}$$

for $\delta > 0$. Letting $\alpha \to \infty$, we then have

$$\tilde{u}_\infty(y) \geq \int_{M \setminus B_{x_0}(\delta)} G_\infty(x_0; z) u_\infty(z)^{2^\# - 1} dv_g(z) \tag{IV.16}$$

using dominated convergence, where G_∞ is the Green's function for the operator $\Delta_g^2 + b\Delta_g + c$. Taking $\delta \to 0$, we obtain

$$\tilde{u}_\infty(y) \geq \int_M G_\infty(x_0; z) u_\infty(z)^{2^\# - 1} dv_g(z) = u_\infty(x_0) \tag{IV.17}$$

using Green's representation on u_∞ solving its own limiting equation. But then as we already said for $|y| \to \infty$, $\tilde{u}_\infty(y) \to K_0$, and since $u_\infty > 0$ in M, we must have $K_0 > 0$.

Putting this whole discussion together, it leads to the following result.

Proposition 1.3. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, and $(b_\alpha), (c_\alpha)$ sequences of real numbers converging respectively to b and c. Let (u_α) a bounded sequence of non-negative solutions to (IV.1) in $H^2(M)$ and $u_\infty \in H^2(M)$ their weak limit. Then there exist $\delta > 0$, $A \geq 0$ and a positive bi-harmonic function $\varphi \in C^4(B_0(2\delta))$ such that up to a subsequence,

$$\tilde{u}_\alpha(y) \to \frac{A}{|y|^{n-4}} + \varphi(y) \quad \text{ in } C^3_{loc}(B_0(2\delta) \setminus \{0\}), \tag{IV.18}$$

where $\tilde{u}_\alpha$ is defined as in (IV.6). Moreover, if $u_\infty \not\equiv 0$, then $\varphi > 0$.

Proof. We have almost all the necessary steps to prove our claim. By the discussion we can take $\delta > 0$ such that $B_0(3\delta)$ only intersects $\tilde{\mathcal{S}}$ at $\{0\}$, then by Lemma 1.2, up to a subsequence $\tilde{u}_\alpha \to \tilde{u}_\infty$ in $C^3_{loc}(B_0(2\delta) \setminus \{0\})$, and by (IV.14) we know that

$$\tilde{u}_\infty(y) = \frac{A}{|y|^{n-4}} + \varphi(y) \quad \forall\, x \in B_0(2\delta) \setminus \{0\}, \tag{IV.19}$$

with $A > 0$ and φ bi-harmonic in $B_0(2\delta)$. We know its expression, $\varphi \geq 0$,

$$\varphi(y) = \sum_{i=1}^{L} \left(\frac{b_i}{|y - y_i|^{n-4}} + \frac{a_i}{|y - y_i|^{n-2}} \right) + K_0 \geq 0, \tag{IV.20}$$

and if $u_\infty \not\equiv 0$, $K_0 > 0$ so that $\varphi > 0$. □

Remark : Under this zoom at a distance of the interaction radius of the biggest bubble, $\tilde{u}_\alpha$ then tends to $\tilde{u}_\infty$ which has the contribution of its own bubble

$$B_\alpha(\exp_{x_\alpha}(\sqrt{\mu_\alpha} y)) = \left(\frac{\mu_\alpha}{\mu_\alpha^2 + \frac{\mu_\alpha |y|^2}{\sqrt{\lambda_n}}} \right)^{\frac{n-4}{2}} \xrightarrow{\alpha \to \infty} \frac{\lambda_n^{\frac{n-4}{4}}}{|y|^{n-4}} \tag{IV.21}$$

but also the contribution of all the other bubbles whose centers were not sent to infinity, and finally, when non-zero, the contribution of $u_\infty \not\equiv 0$ that comes as a global constant K_0.

We then have the following Lemma, that links the local growth of the L^2-norm of the functions with the one of their gradient inside balls of appropriate interaction radius.

Lemma 1.4. Let $\delta > 0$ be given by the Proposition 1.3, then

$$\frac{\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} u_\alpha^2 dv_g}{\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 \, dv_g} \to 0 \quad \text{for } \alpha \to \infty. \tag{IV.22}$$

Proof. First of all we have

$$\int_{B_0(\delta)} \tilde{u}_\alpha^2 dx = \int_{B_0(r)} \tilde{u}_\alpha^2 dx + \int_{B_0(\delta)\setminus B_0(r)} \tilde{u}_\alpha^2 dx \tag{IV.23}$$

for $r > 0$. But in one hand by the last Proposition 1.3, we know that

$$\int_{B_0(\delta)\setminus B_0(r)} \tilde{u}_\alpha^2 dx \leq C \tag{IV.24}$$

is uniformly bounded, while we also know

$$\begin{aligned} \int_{B_0(r)} \tilde{u}_\alpha^2 dx &\leq \left(\int_{B_0(r)} \tilde{u}_\alpha^{2^*} dx \right)^{2/2^*} \mathrm{Vol}_g(B_0(r))^{\frac{2^*-2}{2^*}}, \qquad \text{with } 2^* := \frac{2n}{n-2} \\ &\leq C \left(\int_{B_0(\delta)} \tilde{u}_\alpha^{2^*} dx \right)^{2/2^*} (r^n)^{\frac{2}{n}}. \end{aligned} \tag{IV.25}$$

We then use that $H^1(M) \hookrightarrow L^{2^*}(M)$ continuously, so that

$$\int_{B_0(r)} \tilde{u}_\alpha^2 dx \leq C \int_{B_0(\delta)} |\nabla \tilde{u}_\alpha|^2 \, dx, \tag{IV.26}$$

fixing a small $r > 0$, with C constant independent of α and δ. This gives

$$\int_{B_0(\delta)} \tilde{u}_\alpha^2 dx \leq C + C' \int_{B_0(\delta)} |\nabla \tilde{u}_\alpha|^2 \, dx. \tag{IV.27}$$

Coming back to $\tilde{u}_\alpha(y) = u_\alpha(\exp_{x_\alpha}(\sqrt{\mu_\alpha} y))$, by substitution,

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} u_\alpha^2 dv_g \leq C\mu_\alpha^{\frac{n}{2}} + C'\mu_\alpha \int_{B_{x_0}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 \, dv_g. \tag{IV.28}$$

In the following, we use the fact that $|\nabla u_\alpha| \in H^1(M) \hookrightarrow L^{2^*}(M)$, with $2^* = \frac{2n}{n-2}$ critical exponent. We know by the H^2-decomposition of u_α that

$$\nabla u_\alpha = \nabla u_\infty + \sum_{i=1}^{K} \nabla B_\alpha^i + R_\alpha, \tag{IV.29}$$

with $|R_\alpha|_g \to 0$ in $H^1(M)$. In this function space, the proper scaling and normalizing operation is given by

$$\overline{R}_x^\mu u(y) := \mu^{\frac{n-2}{2}} u(\exp_x(\mu y)), \tag{IV.30}$$

that leaves both the D^1-norm $\|\nabla\cdot\|_{L^2}$ and the L^{2^*}-norm unchanged.

Let us take ϑ cutoff function, that is $\vartheta\in C_0^\infty(\mathbb{R}^n)$ such that $0\leq\vartheta\leq 1$ and

$$\begin{cases}\vartheta\equiv 1 & \text{in } B_0(1/2)\\ \vartheta\equiv 0 & \text{in } \mathbb{R}^n\setminus B_0(1),\end{cases} \tag{IV.31}$$

and ϑ_α such that $\overline{R}_{x_\alpha}^{\mu_\alpha}\vartheta_\alpha=\vartheta$. That means that

$$\vartheta_\alpha(x)=\mu_\alpha^{-\frac{n-2}{2}}\vartheta(\frac{1}{\mu_\alpha}\exp_{x_\alpha}^{-1}(x)) \tag{IV.32}$$

is a cutoff in M around x_α. Fixing $r>0$, for a big enough α we have $r\sqrt{\mu_\alpha}>\mu_\alpha$ so that

$$\int_{B_{x_\alpha}(r\sqrt{\mu_\alpha})}|\nabla u_\alpha|^2\,dv_g\geq\int_{B_{x_\alpha}(\mu_\alpha)}|\nabla u_\alpha|^2\,dv_g\geq\mu_\alpha^2\int_{B_{x_\alpha}(\mu_\alpha)}|\nabla u_\alpha|^2\,\vartheta_\alpha^{2^*-2}dv_g. \tag{IV.33}$$

Using the decomposition (IV.29), one can compute that

$$\int_{B_{x_\alpha}(\mu_\alpha)}|\nabla u_\alpha|^2\,\vartheta_\alpha^{2^*-2}dv_g=\int_{B_{x_\alpha}(\mu_\alpha)}|\nabla B_\alpha|^2\,\vartheta_\alpha^{2^*-2}dv_g+o(1), \tag{IV.34}$$

where $o(1)\to 0$ as $\alpha\to\infty$ and B_α is the bubble centered at x_α and with weight μ_α. Carrying out the zoom and normalization operation, we use that

$$\overline{R}_{x_\alpha}^{\mu_\alpha}\nabla B_\alpha=\eta\nabla B \tag{IV.35}$$

and thus for α big enough

$$\int_{B_{x_\alpha}(\mu_\alpha)}|\nabla B_\alpha|^2\,\vartheta_\alpha^{2^*-2}dv_g=\int_{B_0(1)}|\nabla B|^2\,\vartheta^{2^*-2}dv_{g_\alpha} \tag{IV.36}$$

where $g_\alpha(x)=\left(\exp_{x_\alpha}^*g\right)(\mu_\alpha x)$. This last integral being a fixed real number, when we put together (IV.33–IV.36) we must have for all $r>0$, and α big enough, that there is $C>0$ independent of α such that

$$\int_{B_{x_\alpha}(r\sqrt{\mu_\alpha})}|\nabla u_\alpha|^2\,dv_g\geq C\mu_\alpha^2. \tag{IV.37}$$

Choosing $r=\delta$, we come back to (IV.28) and we have finally

$$\begin{aligned}\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})}u_\alpha^2dv_g&\leq C\mu_\alpha^2\mu_\alpha^{\frac{n-4}{2}}+C'\mu_\alpha\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})}|\nabla u_\alpha|^2\,dv_g\\&\leq C(\mu_\alpha^{\frac{n-4}{2}}+\mu_\alpha)\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})}|\nabla u_\alpha|^2\,dv_g,\end{aligned} \tag{IV.38}$$

which concludes the proof. □

Remark : We also see that we can use (IV.37) to obtain that

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})}|\nabla u_\alpha|^2\,dv_g\geq C\mu_\alpha^2 \tag{IV.39}$$

for all α, where $C>0$ is independent of α.

Proving the first stability Theorem 1.1. Let us recall Equation (IV.2) of the conformal transformation of the Q-curvature,

$$\Delta_g^2 u - \mathrm{div}_g(A_g(\nabla u)) + \frac{n-4}{2} Q_g u = \frac{n-4}{2} Q_{\tilde{g}} u^{2^\#-1}, \tag{IV.40}$$

with

$$A_g = \frac{(n-2)^2+4}{2(n-1)(n-2)} Sc_g \; g - \frac{4}{n-2} Ricc_g \quad \text{bi-linear tensor,} \tag{IV.41}$$

if $\tilde{g} = u^{\frac{4}{n-4}} g$. In a conformally flat Riemannian manifold, one can find $\upsilon \in C^4(M)$ such that $g = \upsilon^{\frac{4}{n-4}} \xi$ and ξ the Euclidean flat metric. In that case $Q_\xi = 0$ and $Pan_\xi u = (-\Delta)^2 u$, with conformal transformation

$$Pan_g u = \upsilon^{-\frac{n+4}{n-4}} Pan_\xi(\upsilon u). \tag{IV.42}$$

Now if $(u_\alpha)_\alpha$ is a bounded sequence in $H^2(M)$ of solutions to IV.1, we have

$$\begin{aligned} P_g u_\alpha &= \Delta_g^2 u_\alpha + b_\alpha \Delta_g u_\alpha + c_\alpha u_\alpha = u_\alpha^{2^\#-1} \\ &= Pan_g u_\alpha - \mathrm{div}_g((b_\alpha g - A_g)\nabla u_\alpha) + (c_\alpha - \frac{n-4}{2} Q_g) u_\alpha. \end{aligned} \tag{IV.43}$$

We write $\hat{u}_\alpha := \upsilon u_\alpha$, then by multiplying the equation above by $\upsilon^{\frac{n+4}{n-4}} = \upsilon^{2^\#-1}$,

$$(-\Delta)^2 \hat{u}_\alpha - \upsilon^{2^\#-1} \mathrm{div}_g((b_\alpha g - A_g)\nabla(\upsilon^{-1}\hat{u}_\alpha)) + (c_\alpha - \frac{n-4}{2} Q_g)\upsilon^{2^\#-2}\hat{u}_\alpha = \hat{u}_\alpha^{2^\#-1}. \tag{IV.44}$$

We can then expand these terms,

$$\begin{aligned} -\mathrm{div}_g(\nabla(\upsilon^{-1}\hat{u}_\alpha)) &= \Delta_g(\upsilon^{-1})\hat{u}_\alpha + 2\upsilon^{-2}\left(\nabla\upsilon, \nabla\hat{u}_\alpha\right)_g + \upsilon^{-1}\Delta_g\hat{u}_\alpha \\ &= \upsilon^{-\frac{n+2}{n-4}}(-\Delta)(\upsilon^{\frac{2}{n-4}})\hat{u}_\alpha - \frac{n-2}{4(n-1)} Sc_g \upsilon^{-1}\hat{u}_\alpha \\ &\quad + 2\upsilon^{-2}\upsilon^{-\frac{4}{n-4}}\left(\nabla\upsilon, \nabla\hat{u}_\alpha\right)_\xi \\ &\quad + \upsilon^{-\frac{2n-2}{n-4}}(-\Delta)(\upsilon^{\frac{n-2}{n-4}}\hat{u}_\alpha) - \frac{n-2}{4(n-1)} Sc_g \upsilon^{-1}\hat{u}_\alpha \end{aligned} \tag{IV.45}$$

using that under this conformal transformation $g = \left(\upsilon^{\frac{n-2}{n-4}}\right)^{\frac{4}{n-2}} \xi$, the Laplacian transforms as

$$\Delta_g\varphi + \frac{n-2}{4(n-1)} Sc_g\varphi = \left(\upsilon^{\frac{n-2}{n-4}}\right)^{-\frac{n+2}{n-2}} (-\Delta)(\upsilon^{\frac{n-2}{n-4}}\varphi). \tag{IV.46}$$

Finally, since

$$\begin{aligned} &(-\Delta)(\upsilon^{\frac{n-2}{n-4}}\hat{u}_\alpha) = \upsilon^{\frac{n-2}{n-4}}(-\Delta)\hat{u}_\alpha - \frac{2(n-2)}{n-4}\upsilon^{\frac{2}{n-4}}\left(\nabla\upsilon, \nabla\hat{u}_\alpha\right)_\xi + \hat{u}_\alpha(-\Delta)(\upsilon^{\frac{n-2}{n-4}}) \\ &-\mathrm{div}_g(A_g\nabla(\upsilon^{-1}\hat{u}_\alpha)) = -\mathrm{div}_g(\upsilon^{-1}A_g\nabla\hat{u}_\alpha) - \hat{u}_\alpha\,\mathrm{div}_g(A_g\nabla(\upsilon^{-1})) + \upsilon^{-2}A_g(\nabla\hat{u}_\alpha, \nabla\upsilon), \end{aligned} \tag{IV.47}$$

we get the equation for $\hat{u}_\alpha$

$$\Delta^2\hat{u}_\alpha + b_\alpha \upsilon^{\frac{4}{n-4}}(-\Delta)\hat{u}_\alpha + B_\alpha(\nabla\upsilon,\nabla\hat{u}_\alpha) + C_\alpha\hat{u}_\alpha \\ + \upsilon^{\frac{n+4}{n-4}}\operatorname{div}_g(A_g\nabla(\upsilon^{-1}))\hat{u}_\alpha + \upsilon^{\frac{n+4}{n-4}}\operatorname{div}_g(\upsilon^{-1}A_g\nabla\hat{u}_\alpha) = \hat{u}_\alpha^{2^\#-1} \quad \text{(IV.48)}$$

with

$$\begin{aligned} B_\alpha &:= -\frac{4}{n-4}b_\alpha\upsilon^{\frac{8-n}{n-4}}\xi - \upsilon^{\frac{12-n}{n-4}}A_g \\ C_\alpha &:= \upsilon^{\frac{2}{n-4}}(-\Delta)(\upsilon^{\frac{2}{n-2}}) - \frac{n-2}{2(n-1)}Sc_g\upsilon^{\frac{8}{n-4}} + \upsilon^{\frac{6-n}{n-4}}(-\Delta)(\upsilon^{\frac{n-2}{n-4}}) + \upsilon^{\frac{8}{n-4}}\left(c_\alpha - \frac{n-4}{2}Q_g\right). \end{aligned} \quad \text{(IV.49)}$$

We also have a Pohozaev-type identity for fourth-order bi-Laplacian operators, if $\Omega \subset \mathbb{R}^n$ is a smooth bounded set, then for all $u \in C^4(\overline{\Omega})$,

$$\int_\Omega (x^k\partial_k u)\Delta^2 u\, dx + \frac{n-4}{2}\int_\Omega u\Delta^2 u\, dx = \frac{n-4}{2}\int_{\partial\Omega}\left(-u\frac{\partial\Delta u}{\partial\nu} + \frac{\partial u}{\partial\nu}\Delta u\right)d\sigma \\ + \int_{\partial\Omega}\left(\frac{1}{2}(x,\nu)(\Delta u)^2 - (x,\nabla u)\frac{\partial\Delta u}{\partial\nu} + \frac{\partial(x,\nabla u)}{\partial\nu}\Delta u\right)d\sigma \quad \text{(IV.50)}$$

where ν is the outwards pointing normal vector of the boundary $\partial\Omega$ and $d\sigma$ its infinitesimal surface element. This identity linking fourth-order differentiation with third-order on the boundary allows us to get estimates on the remaining leading term $(A_g - b_\alpha g)$.

Remark : Generally, Pohozaev-type identities allow to highlight the lower-order terms remaining after performing a certain transformation. Here we know that the Euclidean bubble has a certain scaling invariance, this particular integral (IV.50) in the left-hand term aims to remove all the terms that do transform as the bubble, for which one can verify that it gets 0. In our setting of sequences of solutions, we will see that a lot of terms indeed do cancel out, and we remain with the lower order terms in the right-hand side.

Proposition 1.5. Let (M,g) be a compact locally conformally flat Riemannian manifold of dimension $n \geq 5$, and $b, c > 0$ real constants such that $\frac{b^2}{4} - c > 0$. If $(b_\alpha), (c_\alpha)$ are converging sequences to b and c respectively, and $(u_\alpha)_\alpha$ is a bounded sequence in $H^2(M)$ of positive, non-trivial solutions to IV.1 that blow up. Then $\exists\ \delta > 0$ and $K(u_\infty) \geq 0$ such that

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})}(A_g - b_\alpha g)(\nabla u_\alpha,\nabla u_\alpha)dv_g = o\left(\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})}|\nabla u_\alpha|^2\, dv_g\right) - (K(u_\infty)+o(1))\,\mu_\alpha^{\frac{n-4}{2}},$$

where A_g has the expression (IV.41), $\mu_\alpha = \max_{i=1,\dots K}\mu_{i,\alpha}$ and is x_α the associated blow-up center. Moreover, if $u_\infty \not\equiv 0$, then $K(u_\infty) > 0$.

Proof. We use the rescaling of u_α at the interaction radius, $\tilde{u}_\alpha(y) = u_\alpha(\exp_{x_\alpha}(\sqrt{\mu_\alpha}y))$, Proposition 1.3 gives us $\delta > 0$, $A > 0$ and a bi-harmonic $\varphi \in C^4(B_0(2\delta))$ such that

$$\tilde{u}_\alpha(y) \to \tilde{u}_\infty(y) := \frac{A}{|y|^{n-4}} + \varphi(y) \qquad \text{in } C^3_{loc}(B_0(2\delta)\setminus\{0\}), \quad \text{(IV.51)}$$

with $\varphi \geq 0$ and even $\varphi > 0$ when $u_\infty \not\equiv 0$.

We then write x_∞ the limit of x_α, and we take $\delta_0 > 0$ and a metric $\hat{g}$ which is flat in $B_{x_\infty}(4\delta_0)$, with $g = \upsilon^{\frac{4}{n-4}}\hat{g}$ (which exists since M is locally conformally flat). We can assume without loss of generality that $\upsilon(x_\infty) = 1$, and we write $\hat{u}_\alpha = \upsilon u_\alpha$. By abuse of notation, we will identify x_α with 0 in $B_{x_\alpha}(\delta\sqrt{\mu_\alpha})$ through the exponential map $\exp_{x_\alpha}$, for a small enough $\delta_0 > \delta > 0$. All our functions will then be regarded as defined on the Euclidean space, still writing υ instead of $\upsilon \circ \exp_{x_\alpha}$, A_g for $(\exp_{x_\alpha})^* A_g$ and $\hat{g}$ for $\exp_{x_\alpha}^* \hat{g} = \xi$. The Pohozaev identity can then be applied on these $\hat{u}_\alpha$ on the domain $B_0(\delta\sqrt{\mu_\alpha})$, we get

$$\begin{aligned}
LHS_\alpha :=& \int_{B_0(\delta\sqrt{\mu_\alpha})} (x^k \partial_k \hat{u}_\alpha)\Delta^2 \hat{u}_\alpha dx + \frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha \Delta^2 \hat{u}_\alpha dx \\
=& \frac{n-4}{2}\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \left(-\hat{u}_\alpha \frac{\partial \Delta \hat{u}_\alpha}{\partial \nu} + \frac{\partial \hat{u}_\alpha}{\partial \nu}\Delta \hat{u}_\alpha\right) d\sigma \\
&+ \int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \left(\frac{1}{2}(x,\nu)(\Delta \hat{u}_\alpha)^2 - (x, \nabla \hat{u}_\alpha)\frac{\partial \Delta \hat{u}_\alpha}{\partial \nu} + \frac{\partial (x, \nabla \hat{u}_\alpha)}{\partial \nu}\Delta \hat{u}_\alpha\right) d\sigma \\
=& (\sqrt{\mu_\alpha})^{n-4}\Big[\frac{n-4}{2}\int_{|y|=\delta} \left(-\hat{\hat{u}}_\alpha \frac{\partial \Delta \hat{\hat{u}}_\alpha}{\partial \nu} + \frac{\partial \hat{\hat{u}}_\alpha}{\partial \nu}\Delta \hat{\hat{u}}_\alpha\right) d\sigma \\
&+ \int_{|y|=\delta} \left(\frac{1}{2}(y,\nu)(\Delta \hat{\hat{u}}_\alpha)^2 - \left(y, \nabla \hat{\hat{u}}_\alpha\right)\frac{\partial \Delta \hat{\hat{u}}_\alpha}{\partial \nu} + \frac{\partial \left(y, \nabla \hat{\hat{u}}_\alpha\right)}{\partial \nu}\right)\Delta \hat{\hat{u}}_\alpha d\sigma\Big]
\end{aligned} \tag{IV.52}$$

by the change of variable $x = \sqrt{\mu_\alpha} y$, where $\hat{\hat{u}}_\alpha := \hat{u}_\alpha(\sqrt{\mu_\alpha} y)$. One sees then that by definition

$$\hat{\hat{u}}_\alpha(y) = \upsilon(\sqrt{\mu_\alpha} y) u_\alpha(\exp_{x_\alpha}(\sqrt{\mu_\alpha} y)) \to \tilde{u}_\infty(y) = \frac{A}{|y|^{n-4}} + \varphi(y) \tag{IV.53}$$

in $C^3_{loc}(B_0(2\delta) \setminus \{0\})$ since $\upsilon(0) = 1$, with $\Delta^2 \varphi = 0$.

For all $\delta > r > 0$ we know also by the Pohozaev identity used on the domain $\Omega = B_0(\delta) \setminus B_0(r)$ and the fact that $\tilde{u}_\infty$ is biharmonic on Ω that

$$\begin{aligned}
&\frac{n-4}{2}\int_{|x|=\delta} \left(-\tilde{u}_\infty \frac{\partial \Delta \tilde{u}_\infty}{\partial \nu} + \frac{\partial \tilde{u}_\infty}{\partial \nu}\Delta \tilde{u}_\infty\right) d\sigma + \int_{|x|=\delta} \Big(\frac{1}{2}(x,\nu)(\Delta \tilde{u}_\infty)^2 \\
&\qquad - (x, \nabla \tilde{u}_\infty)\frac{\partial \Delta \tilde{u}_\infty}{\partial \nu} + \frac{\partial (x, \nabla \tilde{u}_\infty)}{\partial \nu}\Delta \tilde{u}_\infty\Big) d\sigma \\
=& \frac{n-4}{2}\int_{|x|=r} \left(-\tilde{u}_\infty \frac{\partial \Delta \tilde{u}_\infty}{\partial \nu} + \frac{\partial \tilde{u}_\infty}{\partial \nu}\Delta \tilde{u}_\infty\right) d\sigma + \int_{|x|=r} \Big(\frac{1}{2}(x,\nu)(\Delta \tilde{u}_\infty)^2 \\
&\qquad - (x, \nabla \tilde{u}_\infty)\frac{\partial \Delta \tilde{u}_\infty}{\partial \nu} + \frac{\partial (x, \nabla \tilde{u}_\infty)}{\partial \nu}\Delta \tilde{u}_\infty\Big) d\sigma \\
&\xrightarrow{r \to 0} CA\varphi(0) =: K(u_\infty),
\end{aligned} \tag{IV.54}$$

letting $r \to 0$, by the expression of $\tilde{u}_\infty$. This gives that $\mu_\alpha^{-\frac{n-4}{2}} LHS_\alpha \to K(u_\infty)$, with $A > 0$ given in (IV.51) and we know $\varphi(0) > 0$ if $u_\infty \not\equiv 0$ so that $K(u_\infty) \geq 0$ with strict positivity if $u_\infty \not\equiv 0$.

Now by the discussion we had before, we get that since on $B_{x_\infty}(4\delta_0)$, $\Delta_{\hat{g}} = -\Delta$ is the Euclidean Laplacian and $\mathrm{div}_{\hat{g}} = \mathrm{div}_\xi$, $\hat{u}_\alpha$ solves the equation (IV.48)

$$\Delta^2\hat{u}_\alpha + b_\alpha \upsilon^{\frac{4}{n-4}}(-\Delta)\hat{u}_\alpha + B_\alpha(\nabla\upsilon, \nabla\hat{u}_\alpha) + C_\alpha\hat{u}_\alpha \\ + \upsilon^{\frac{n+4}{n-4}}\,\mathrm{div}_g(A_g\nabla(\upsilon^{-1}))\hat{u}_\alpha + \upsilon^{\frac{n+4}{n-4}}\,\mathrm{div}_g(\upsilon^{-1}A_g\nabla\hat{u}_\alpha) = \hat{u}_\alpha^{2^\#-1} \quad \text{(IV.55)}$$

on $B_{x_\infty}(4\delta)$, with B_α, C_α defined as in (IV.49). This equation will enable us to compute differently the left-hand side in (IV.50). We have

$$LHS_\alpha = \int_{B_0(\delta\sqrt{\mu_\alpha})} \left(x^k\partial_k\hat{u}_\alpha + \frac{n-4}{2}\hat{u}_\alpha\right)\Big(\hat{u}_\alpha^{2^\#-1} - b_\alpha\upsilon^{\frac{4}{n-4}}(-\Delta)\hat{u}_\alpha \\ - B_\alpha\left(\nabla\upsilon, \nabla\hat{u}_\alpha\right) - C_\alpha\hat{u}_\alpha - \upsilon^{\frac{n+4}{n-4}}\,\mathrm{div}_g(A_g\nabla(\upsilon^{-1}))\hat{u}_\alpha - \upsilon^{\frac{n+4}{n-4}}\,\mathrm{div}_g(\nu^{-1}A_g\nabla\hat{u}_\alpha)\Big)dx, \quad \text{(IV.56)}$$

and by integration by parts

$$\int_{B_0(\delta\sqrt{\mu_\alpha})} x^k\partial_k\hat{u}_\alpha\hat{u}_\alpha^{2^\#-1}dx = -\frac{n-4}{2n}\int_{B_0(\delta\sqrt{\mu_\alpha})}\partial_k x^k\hat{u}_\alpha^{2^\#}dx + \frac{n-4}{2n}\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}(x,\nu)\,\hat{u}_\alpha^{2^\#}d\sigma. \quad \text{(IV.57)}$$

The first term exactly compensates the term $\frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^{2^\#}$ so that

$$\int_{B_0(\delta\sqrt{\mu_\alpha})}\left(x^k\partial_k\hat{u}_\alpha + \frac{n-4}{2}\hat{u}_\alpha\right)(\hat{u}_\alpha^{2^\#-1}) = O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^{2^\#}dx\right). \quad \text{(IV.58)}$$

Similarly, we compute, still using the Einstein summation over repeated indices for simplicity, on one hand

$$\int_{B_0(\delta\sqrt{\mu_\alpha})} x^k\partial_k\hat{u}_\alpha\upsilon^{\frac{4}{n-4}}\Delta\hat{u}_\alpha dx = -\int_{B_0(\delta\sqrt{\mu_\alpha})}\delta_i^k\partial_k\hat{u}_\alpha\upsilon^{\frac{4}{n-4}}\partial_i\hat{u}_\alpha dx - \int_{B_0(\delta\sqrt{\mu_\alpha})} x^k\partial_i\partial_k\hat{u}_\alpha\upsilon^{\frac{4}{n-4}}\partial_i\hat{u}_\alpha dx \\ - \int_{B_0(\delta\sqrt{\mu_\alpha})} x^k\frac{4}{n-4}\partial_k\hat{u}_\alpha\upsilon^{\frac{8-n}{n-4}}\partial_i\upsilon\partial_i u_\alpha dx + \int_{\partial B_0(\delta\sqrt{\mu_\alpha})} x^k\partial_k\hat{u}_\alpha\upsilon^{\frac{4}{n-4}}\partial_i\hat{u}_\alpha\nu^i d\sigma \\ = -\int_{B_0(\delta\sqrt{\mu_\alpha})}\upsilon^{\frac{4}{n-4}}\left|\nabla\hat{u}_\alpha\right|_\xi^2 dx - \int_{B_0(\delta\sqrt{\mu_\alpha})} x^k\upsilon^{\frac{4}{n-4}}\partial_i\partial_k\hat{u}_\alpha\partial_i\hat{u}_\alpha dx \\ - \frac{4}{n-4}\int_{B_0(\delta\sqrt{\mu_\alpha})} x^k\upsilon^{\frac{8-n}{n-4}}\partial_i\upsilon\partial_k\hat{u}_\alpha\partial_i\hat{u}_\alpha dx + O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\left|\nabla\hat{u}_\alpha\right|^2 dx\right)$$

and also

$$\frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha\upsilon^{\frac{4}{n-4}}\Delta\hat{u}_\alpha dx = -\frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})}\partial_i\hat{u}_\alpha\upsilon^{\frac{4}{n-4}}\partial_i\hat{u}_\alpha dx \\ - \frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})}\frac{4}{n-4}\hat{u}_\alpha\upsilon^{\frac{8-n}{n-4}}\partial_i\upsilon\partial_i\hat{u}_\alpha dx + \frac{n-4}{2}\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha\upsilon^{\frac{4}{n-4}}\partial\hat{u}_\alpha\nu^i d\sigma. \quad \text{(IV.59)}$$

With

$$\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha \upsilon^{\frac{4}{n-4}} \partial_i \hat{u}_\alpha \nu^i d\sigma = -\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \partial_i \hat{u}_\alpha \upsilon^{\frac{4}{n-4}} \hat{u}_\alpha \nu^i d\sigma - \int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \frac{4}{n-4} \hat{u}_\alpha^2 \upsilon^{\frac{8-n}{n-4}} \partial_i \upsilon \nu^i d\sigma$$

so that $$\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha \upsilon^{\frac{4}{n-4}} \partial_i \hat{u}_\alpha \nu^i d\sigma = O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha^2 dx\right)$$

and thus finally
$$\frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha \upsilon^{\frac{4}{n-4}} \Delta \hat{u}_\alpha dx = -\frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})} \upsilon^{\frac{4}{n-4}} |\nabla \hat{u}_\alpha|_\xi^2 \, dx$$
$$- 2\int_{B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha \upsilon^{\frac{8-n}{n-4}} \partial_i \upsilon \partial_i \hat{u}_\alpha dx + O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha^2 dx\right), \tag{IV.60}$$

so that some terms cancel with the first part of the integral against B_α :

$$\int_{B_0(\delta\sqrt{\mu_\alpha})} \left(x^k \partial_k \hat{u}_\alpha + \frac{n-4}{2}\hat{u}_\alpha\right) \frac{4}{n-4} b_\alpha \upsilon^{\frac{8-n}{n-4}} \xi(\nabla \upsilon, \nabla \hat{u}_\alpha) dx$$
$$= b_\alpha \frac{4}{n-4}\int_{B_0(\delta\sqrt{\mu_\alpha})} x^k \partial_k \hat{u}_\alpha \upsilon^{\frac{8-n}{n-4}} \partial_i \upsilon \partial_i \hat{u}_\alpha dx + 2\int_{B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha \upsilon^{\frac{8-n}{n-4}} \partial_i \hat{u}_\alpha \partial_i \upsilon dx, \tag{IV.61}$$

and we end up with

$$\int_{B_0(\delta\sqrt{\mu_\alpha})} \left(x^k \partial_k \hat{u}_\alpha + \frac{n-4}{2}\hat{u}_\alpha\right) \left(b_\alpha \upsilon^{\frac{4}{n-4}} \Delta \hat{u}_\alpha + \frac{4}{n-4} b_\alpha \upsilon^{\frac{8-n}{n-4}} \xi(\nabla \upsilon, \nabla \hat{u}_\alpha)\right) dx$$
$$= -\frac{n-2}{2} b_\alpha \int_{B_0(\delta\sqrt{\mu_\alpha})} \upsilon^{\frac{4}{n-4}} |\nabla \hat{u}_\alpha|_\xi^2 \, dx - b_\alpha \int_{B_0(\delta\sqrt{\mu_\alpha})} x^k \upsilon^{\frac{4}{n-4}} \partial_i \partial_k \hat{u}_\alpha \partial_i \hat{u}_\alpha dx$$
$$+ O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} |\nabla \hat{u}_\alpha|^2 \, dx\right) + O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha^2 dx\right). \tag{IV.62}$$

But now operating the other change of variable in the second term

$$\int_{B_0(\delta\sqrt{\mu_\alpha})} x^k \upsilon^{\frac{4}{n-4}} \partial_i \partial_k \hat{u}_\alpha \partial_i \hat{u}_\alpha dx = \frac{1}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})} x^k \upsilon^{\frac{4}{n-4}} \partial_k (|\nabla \hat{u}_\alpha|_\xi^2) dx$$
$$= -\frac{1}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})} \partial_k x^k \upsilon^{\frac{4}{n-4}} |\nabla \hat{u}_\alpha|_\xi^2 \, dx - \frac{1}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})} x^k \frac{4}{n-4} \upsilon^{\frac{8-n}{n-4}} \partial_k \upsilon |\nabla \hat{u}_\alpha|_\xi^2 \, dx$$
$$+ \frac{1}{2}\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} x^k \upsilon^{\frac{4}{n-4}} |\nabla \hat{u}_\alpha|^2 \nu_k d\sigma \tag{IV.63}$$
$$= -\frac{n}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})} \upsilon^{\frac{4}{n-4}} |\nabla \hat{u}_\alpha|_\xi^2 \, dx + O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} |\nabla \hat{u}_\alpha|^2 \, dx\right)$$

by symmetry. The first terms of (IV.56) then become

$$\begin{aligned}\int_{B_0(\delta\sqrt{\mu_\alpha})}&\left(x^k\partial_k\hat{u}_\alpha+\frac{n-4}{2}\hat{u}_\alpha\right)\left(\hat{u}_\alpha^{2^\#-1}+b_\alpha\upsilon^{\frac{4}{n-4}}\Delta\hat{u}_\alpha+\frac{4}{n-4}b_\alpha\upsilon^{\frac{8-n}{n-4}}\xi(\nabla\upsilon,\nabla\hat{u}_\alpha)\right)dx\\ =&b_\alpha\int_{B_0(\delta\sqrt{\mu_\alpha})}\upsilon^{\frac{4}{n-4}}\left|\nabla\hat{u}_\alpha\right|_\xi^2dx+O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^{2^\#}dx\right)\\ &+O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\left|\nabla\hat{u}_\alpha\right|^2dx\right)+O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^2dx\right).\end{aligned}\tag{IV.64}$$

With similar computations on the second part of B_α and the other terms involving A_g, one finally has that

$$\begin{aligned}LHS_\alpha=&b_\alpha\int_{B_0(\delta\sqrt{\mu_\alpha})}\upsilon^{\frac{4}{n-4}}\left|\nabla\hat{u}_\alpha\right|_\xi^2dx-\int_{B_0(\delta\sqrt{\mu_\alpha})}\upsilon^{\frac{8}{n-4}}A_g(\nabla\hat{u}_\alpha,\nabla\hat{u}_\alpha)dx\\ &+o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})}\left|\nabla\hat{u}_\alpha\right|_\xi^2dx\right)+O\left(\int_{B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^2dx\right)\\ &+O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^{2^\#}dx\right)+O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\left|\nabla\hat{u}_\alpha\right|^2dx\right)+O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^2dx\right),\end{aligned}\tag{IV.65}$$

since for the remaining terms

$$\begin{aligned}&\int_{B_0(\delta\sqrt{\mu_\alpha})}\left(x^k\partial_k\hat{u}_\alpha+\frac{n-4}{2}\hat{u}_\alpha\right)C_\alpha\hat{u}_\alpha dx\\ &\quad=\frac{1}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})}x^kC_\alpha\partial_k(\hat{u}_\alpha^2)dx+\frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})}C_\alpha\hat{u}_\alpha^2dx\\ &\quad=-\frac{1}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})}nC_\alpha\hat{u}_\alpha^2dx-\frac{1}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})}x^k\partial_kC_\alpha\hat{u}_\alpha^2dx+\frac{1}{2}\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}x^kC_\alpha\hat{u}_\alpha^2\nu_kd\sigma\\ &\qquad\qquad+\frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})}C_\alpha\hat{u}_\alpha^2dx\\ &\quad=O\left(\int_{B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^2dx\right)+O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^2dx\right).\end{aligned}\tag{IV.66}$$

But we already know that by Lemma 1.4,

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})}u_\alpha^2dv_g=o(1)\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})}\left|\nabla u_\alpha\right|^2dv_g,\tag{IV.67}$$

with $o(1)\to0$ as $\alpha\to\infty$, and we recall that except at $y=0$, $u_\alpha(\exp_{x_\alpha}(\sqrt{\mu_\alpha}y))\to\frac{A}{|y|^{n-4}}+\varphi(y)$ so that

$$\begin{aligned}&\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^2dx\to(\sqrt{\mu_\alpha})^{n-1}\int_{|y|=\delta}\upsilon^2\left(\frac{A}{\delta^{n-4}}+\varphi(y)\right)^2dy=o(\mu_\alpha^{\frac{n-4}{2}})\\ &\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^{2^\#}dx\to(\sqrt{\mu_\alpha})^{n-1}\int_{|y|=\delta}\upsilon^{2^\#}\left(\frac{A}{\delta^{n-4}}+\varphi(y)\right)^{2^\#}dy=o(\mu_\alpha^{\frac{n-4}{2}})\end{aligned}\tag{IV.68}$$

since φ, υ are $C^4(M)$, $\frac{n-1}{2} > \frac{n-4}{2}$ always holds. In the same way

$$\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} |\nabla \hat{u}_\alpha|^2 \, dx = o(\mu_\alpha^{\frac{n-4}{2}}) \tag{IV.69}$$

and finally this gives

$$LHS_\alpha = b_\alpha \int_{B_0(\delta\sqrt{\mu_\alpha})} \upsilon^{\frac{4}{n-4}} |\nabla \hat{u}_\alpha|_\xi^2 \, dx - \int_{B_0(\delta\sqrt{\mu_\alpha})} \upsilon^{\frac{8}{n-4}} A_g(\nabla \hat{u}_\alpha, \nabla \hat{u}_\alpha) dx + o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})} |\nabla \hat{u}_\alpha|_\xi^2 \, dx \right) + o(\mu_\alpha^{\frac{n-4}{2}}). \tag{IV.70}$$

One then realizes that $|\nabla \hat{u}_\alpha|_\xi^2 = (\nabla \hat{u}_\alpha, \nabla \hat{u}_\alpha)_\xi = \upsilon^{\frac{4}{n-4}} (\nabla \hat{u}_\alpha, \nabla \hat{u}_\alpha)_g$, and by (IV.54) we have in the end

$$\int_{B_0(\delta\sqrt{\mu_\alpha})} \upsilon^{\frac{8}{n-4}} (b_\alpha g - A_g)(\nabla \hat{u}_\alpha, \nabla \hat{u}_\alpha) dx = (K(u_\infty) + o(1)) \mu_\alpha^{\frac{n-4}{2}} + o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})} |\nabla \hat{u}_\alpha|_\xi^2 \, dx \right) + o(\mu_\alpha^{\frac{n-4}{2}}). \tag{IV.71}$$

Since $\hat{u}_\alpha := \upsilon u_\alpha$, we use (IV.67) and compute with the same arguments as before

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} (b_\alpha g - A_g)(\nabla u_\alpha, \nabla u_\alpha) dv_g = (K(u_\infty) + o(1)) \mu_\alpha^{\frac{n-4}{2}} + o\left(\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 \, dv_g \right) \tag{IV.72}$$

recalling that 0 was identified to x_α by the identity map of the flat metric ξ around this point. Moreover $K(u_\infty) \geq 0$ with $K > 0$ if $u_\infty \not\equiv 0$. □

All the previous results did hold in any dimension $n \geq 5$, we now consider the case when $n = 5, 6, 7$. We present the following result relying on the assumption that $n < 8$ for technical reasons, and which will justify this separation between low and high dimensions in the proof of Theorem 1.1.

Lemma 1.6. Let (M, g) be a smooth compact Riemannian manifold of dimension $n = 5, 6, 7$, let δ be given by the Proposition 1.3, with $(u_\alpha)_\alpha$ a bounded sequence of non-negative solutions to (IV.1). Then

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 \, dv_g = o(\mu_\alpha^{\frac{n-4}{2}}), \tag{IV.73}$$

with $\mu_\alpha = \max_{i=1,\dots K} \mu_{i,\alpha}$ and x_α the associated center of the bubble concentrating the slowest as defined before.

Proof. We split the integral around all the singularities, fixing $R > 0$,

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 \, dv_g \leq \sum_{i=1}^K \int_{B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 \, dv_g + \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha}) \setminus \cup_{i=1}^K B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 \, dv_g. \tag{IV.74}$$

Then using the estimates of Theorem III.2.1 we know that

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\setminus\cup_{i=1}^K B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 \, dv_g$$
$$\leq C\mathrm{Vol}_g(B_{x_\alpha}(\delta\sqrt{\mu_\alpha})) + C\mu_\alpha^{n-4}\sum_{i=1}^K \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\setminus B_{x_{i,\alpha}}(R\mu_\alpha)} d_g(x_{i,\alpha},x)^{2(4-n-1)}dv_g. \quad \text{(IV.75)}$$

But since each function $d_g(x_{i,\alpha},\cdot)^{6-2n}$ is integrable on the set $B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\setminus B_{x_{i,\alpha}}(R\mu_\alpha)$ respectively, we can write

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\setminus B_{x_{i,\alpha}}(R\mu_\alpha)} d_g(x_{i,\alpha},x)^{6-2n}dv_g \leq C(\mu_\alpha), \quad \text{(IV.76)}$$

where $C(\mu_\alpha)$ is some constant depending on μ_α, that can be computed for each n. In particular we always get $\mu_\alpha^{n-4}C(\mu_\alpha) = o(\mu_\alpha^{\frac{n-4}{2}})$. Similarly $\mathrm{Vol}_g(B_{x_\alpha}(\delta\sqrt{\mu_\alpha})) \leq C\mu_\alpha^{\frac{n}{2}} = o(\mu_\alpha^{\frac{n-4}{2}})$.

Finally, using Hölder's inequality and the fact that all the $|\nabla u_\alpha|$ are uniformly bounded in $L^{\frac{2n}{n-2}}$ thanks to the continuous embedding $H^1 \hookrightarrow L^{\frac{2n}{n-2}}$,

$$\int_{B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 \, dv_g \leq C\mu_\alpha^{n(1-\frac{2(n-2)}{2n})} = C\mu_\alpha^2. \quad \text{(IV.77)}$$

Here the fact that $n < 8$ plays a central role, indeed in that case $\frac{n-4}{2} < 2$ and

$$\int_{B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 \, dv_g = O(\mu_\alpha^2) = o(\mu_\alpha^{\frac{n-4}{2}}) \text{ for } \alpha \to \infty, \quad \text{(IV.78)}$$

and that concludes the proof. □

With all of this we are now ready to prove Theorem 1.1.

Proof (of Theorem 1.1). By contradiction, let us suppose that $u_\infty \not\equiv 0$ and that the u_α blow up. In any case, $A_g, b_\alpha g$ are both bounded bi-linear maps, and thus

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} (b_\alpha g - A_g)(\nabla u_\alpha, \nabla u_\alpha)dv_g = O\left(\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 \, dv_g\right). \quad \text{(IV.79)}$$

By the preceding Lemma, put together with Proposition 1.5, in dimensions $n = 5, 6, 7$ we then have

$$\begin{aligned} &\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} (b_\alpha g - A_g)(\nabla u_\alpha, \nabla u_\alpha)dv_g = o(\mu_\alpha^{\frac{n-4}{2}}) = (K(u_\infty) + o(1))\mu_\alpha^{\frac{n-4}{2}} \\ \Leftrightarrow\ &o(\mu_\alpha^{\frac{n-4}{2}}) = K(u_\infty)\mu_\alpha^{\frac{n-4}{2}} \end{aligned} \quad \text{(IV.80)}$$

with $K(u_\infty) > 0$ if $u_\infty \not\equiv 0$. We have reached a contradiction, and thus $u_\infty \equiv 0$ if we suppose that the sequence u_α blows up. □

This result might seem surprising, that in dimensions lower than 8 one can only observe the blowing up of sequences of solutions if their weak limit is zero. If we start with a solution $u_\infty > 0$

such that $P_g u_\infty = u_\infty^{2^\#-1}$, we know that by Proposition II.4.2, bubbles are almost solutions in the sense that they are Palais-Smale sequences for the functional associated to the equation

$$\Delta_g^2 u + b\Delta_g u + cu = u^{2^\#-1}, \tag{IV.81}$$

one gets $P_g B_\alpha = B_\alpha^{2^\#-1} + o(1)$, where $o(1) \to 0$ in $(H^2(M))'$. This previous Theorem tells us that we will never be able to define a slight perturbation $\|R_\alpha\|_{H^2} \to 0$ such that $u_\infty + B_\alpha + R_\alpha$ are solutions to a sequence of equations approaching (IV.81).

Since $u_\infty \not\equiv 0$, a concentrating bubble B_α will only stay dominant pointwise at an interaction radius $\sim \sqrt{\mu_\alpha}$. Zooming at this scale, the bubble far from its center has a behaviour of order $\frac{1}{|x|^{n-4}}$, and is essentially defined on $\mathbb{R}^n$. For $n \leq 8$, it happens that $\frac{1}{x^{n-4}} \notin L^2(\mathbb{R}^n \setminus B_0(1))$, in some sense B_α does not decrease fast enough at infinity so that $u_\infty + B_\alpha$ could be perturbed in a real sequence of solutions to equations of the type

$$\Delta_g^2 u + b_\alpha \Delta_g u + c_\alpha u = u^{2^\#-1}. \tag{IV.82}$$

If $u_\infty \equiv 0$, the bubble remains the only dominant term everywhere on the compact manifold, and Proposition II.4.2 ensures that such functions are asymptotically solution to any equation of the shape $\Delta_g^2 u - \mathrm{div}_g(A\nabla u) + au = u^{2^\#-1}$. In particular, for any sequence of solutions to (IV.82), there could be a sequence of concentrating bubbles and a perturbation $\|R_\alpha\|_{H^2} \to 0$.

The case when $n = 5$. When the dimension of the manifold is exactly 5, if we denote by G the Green's function of the fourth operator we are interested in, $P_g = \Delta_g^2 + b\Delta_g + c$, then we can express

$$G(x, y) = \frac{1}{6\omega_4 d_g(x, y)} + \mu_x(y), \tag{IV.83}$$

where ω_4 is the volume of the 5−sphere, so that the term $\frac{1}{6\omega_4 d_g(x,y)}$ represents the Green's function of the bi-Laplacian operator on $M \setminus \{x\}$, and $\mu_x \in C^0(M)$ is a function depending on the basis point $x \in M$ taking care of the lower order terms.

Definition 1.7. We define the *mass* of the operator P_g at $x \in M$ as the quantity $\mu_x(x)$.

We then have the following result, refining our study of the stability of the equations, depending on this mass function that is intrinsic to the manifold and the particular operator we consider.

Theorem 1.8. Let (M, g) be a compact locally conformally flat manifold of dimension $n = 5$. Let $b, c > 0$ such that $\frac{b^2}{4} - c > 0$. If $\mu_x(x) > 0$ for all $x \in M$, then for every sequence of real numbers $(b_\alpha), (c_\alpha)$ converging respectively to b and c and for each sequence $(u_\alpha)_\alpha$ of smooth non-trivial positive solutions to (IV.82) that is bounded in $H^2(M)$, u_α admits a positive weak limit u_∞ such that $u_\alpha \to u_\infty$ in $C^4(M)$.

In particular, this Theorem ensures the stability of equation (IV.81) even in the case that the only possible solutions are $u_\infty \equiv 0$.

The study of the positivity of a differential operator's mass is a commonly asked question, and is motivated in part by the second-order setting. In the latter case, one can refer to Schoen and Yau (1981) that proved the positivity of the mass for the conformal Laplacian operator, completing resolution of the Yamabe problem on compact Riemannian manifolds (Schoen, 1984). Here we can

rely on results coming from Humbert and Raulot (2009), giving the positivity of the mass for the Paneitz-Branson operator when the manifold is conformally flat, with positive Yamabe invariant and with a positive operator. They showed that in this case the mass is non-negative, and can only be 0 if (M, g) is conformally diffeomorphic to the standard sphere. One can also see Avalos, Laurain, and Lira (2022) that studied the positivity of the mass for fourth-order operators whose interest relies in general relativity, but which can be useful for the Paneitz operator.

Assume $n = 5$, and let $(u_\alpha)_\alpha$ be a bounded sequence in $H^2(M)$ of positive solutions to (IV.82) that blow up. With Theorem 1.1 we know that $u_\alpha \rightharpoonup 0$ weakly in $H^2(M)$ otherwise no bubble could be created. We define the set of concentration points $\mathcal{S} := \{x_1, x_2, \dots x_N\}$, where u_α blows up at the limit, with $N < K$ the total number of bubbles if clusters are created.

In the case of a unique bubble, $\mathcal{S} = \{x_1\}$, and if (u_α) blows up $u_\infty \equiv 0$ so that for all compact subset $\Omega \subseteq M \setminus \{x_1\}$,

$$\frac{\int_\Omega u_\alpha^{2^\#-1} dv_g}{\int_M u_\alpha^{2^\#-1} dv_g} = o(1) \qquad \text{and} \qquad \frac{\int_{B_{x_1}(\delta)} u_\alpha^{2^\#-1} dv_g}{\int_M u_\alpha^{2^\#-1} dv_g} \xrightarrow{\alpha\to\infty} \lambda := 1 \quad \forall\ \delta > 0. \tag{IV.84}$$

Moreover, defining $\tilde{u}_\alpha = \frac{1}{\int_M u_\alpha^{2^\#-1} dv_g} u_\alpha$, since u_α is a solution to (IV.82) then $\tilde{u}_\alpha$ is solution to

$$\Delta_g^2 \tilde{u}_\alpha + b_\alpha \Delta_g \tilde{u}_\alpha + c_\alpha \tilde{u}_\alpha = m_\alpha \tilde{u}_\alpha^{2^\#-1} \tag{IV.85}$$

$$\begin{aligned} \text{with} \quad m_\alpha &:= \left(\int_M u_\alpha^{2^\#-1} dv_g \right)^{2^\#-2} \to \left(\int_{B_0(1)} \left(\frac{\mu_\alpha}{\mu_\alpha^2 + |x|^2 / \sqrt{\lambda_n}} \right)^{\frac{1}{2}} dx \right)^8 \\ &= O(\mu_\alpha^4). \end{aligned} \tag{IV.86}$$

But now by the estimates of Theorem III.2.1 on u_α, for all compact subset $\Omega \subset M \setminus \mathcal{S}$, we have $\tilde{u}_\alpha \leq C$ constant depending on Ω, so that there exists $\tilde{u}_\infty \in C^4(M \setminus \mathcal{S})$ such that $\tilde{u}_\alpha \to \tilde{u}_\infty$ in $C^4_{loc}(M \setminus \mathcal{S})$. Then $\tilde{u}_\infty$ solves the limiting equation

$$\Delta_g^2 \tilde{u}_\infty + b \Delta_g \tilde{u}_\infty + c \tilde{u}_\infty = 0 \qquad \text{in } M \setminus \mathcal{S}, \tag{IV.87}$$

and thus by elliptic Theory, P_g being coercive we know $\tilde{u}_\infty = C G_{x_1}$ for some $C > 0$, where $G_{x_1} = G(x_1, \cdot)$ is the Green's function of the operator $P_g = \Delta_g^2 + b\Delta_g + c$.

Remark : When $d := \frac{b^2}{4} - c > 0$, we know that we can express G as

$$G(x, y) = \int_M G_1(x, z) G_2(z, y) dv_g(z), \tag{IV.88}$$

with G_1, G_2 are the Green's operators of respectively $\Delta_g + \frac{b - \sqrt{d}}{2}$ and $\Delta_g + \frac{b + \sqrt{d}}{2}$. In particular, in dimension $n = 5$, by the expression of G (IV.83) we have $G > 0$ if $\mu_x(x) > 0$.

Moreover, by (IV.84), we have to impose the constant in $\tilde{u}_\infty$, $\tilde{u}_\infty = \lambda G_{x_1}$ so that

$$\tilde{u}_\alpha \to \lambda G_{x_1} \qquad \text{in } C^4_{loc}(M \setminus \mathcal{S}). \tag{IV.89}$$

Now if (M, g) is locally conformally flat, let $\delta_0 > 0$ and take $\upsilon > 0$ a function with $\upsilon(0) = 1$ such that $\upsilon^{\frac{4}{n-4}} \xi = g$ in $B_0(\delta_0)$ seen as a Euclidean ball, identifying x_1 to 0 in $B_{x_1}(\delta_0)$ as in the proof of Proposition 1.5. We define $\hat{u}_\alpha := \upsilon u_\alpha$, since

$$\frac{u_\alpha}{\int_M u_\alpha^{2^\#-1} dv_g} \to \lambda G_{x_1}, \tag{IV.90}$$

and by the expression of G, we get

$$\begin{aligned}\frac{\hat{u}_\alpha(x)}{\int_M u_\alpha^{2^\#-1}dv_g} &\xrightarrow{\alpha\to\infty} \lambda\left(\frac{C}{d_g(0,x)}+\mu_x(x)\right)\left(\upsilon(0)+C_1\left(\nabla\upsilon(0),x\right)+o(|x|)\right)\\ &= \frac{C\lambda}{|x|}+\beta(x).\end{aligned} \tag{IV.91}$$

Then $\beta \in C^0(B_0(\delta_0)\setminus\{0\})$ represents the non-dominating terms as $|x| \ll 1$, with by definition $\lim_{x\to 0}\beta(x) = \lambda\mu_{x_1}(x_1)$.

In the case of N distinct concentration centers only few things need to change, we define

$$\lambda_i := \lim_{\alpha\to\infty}\frac{\int_{B_{x_i}(\delta)} u_\alpha^{2^\#-1}dv_g}{\int_M u_\alpha^{2^\#-1}dv_g} \tag{IV.92}$$

independent of $0<\delta<\delta_0$, fixing $\delta_0 < \inf_{i\neq j} d_g(x_i,x_j)$, with $\lambda_i > 0$ and $\sum_{i=1}^N \lambda_i = 1$. Then by (IV.87),

$$\tilde{u}_\alpha = \frac{u_\alpha}{\int_M u_\alpha^{2^\#-1}dv_g} \to \sum_{i=1}^N \lambda_i G_{x_i}, \tag{IV.93}$$

and for each fixed i we can take $\upsilon^i > 0$ function such that $(\upsilon^i)^{\frac{4}{n-4}}\xi = g$ in $B_{x_i}(\delta_0)$. Once again we will identify 0 to x_i and by supposition $B_{x_i}(\delta_0)\cap\mathcal{S} = \{x_i\}$. Defining $\hat{u}_\alpha^i := \upsilon^i u_\alpha$, by the expression of the Green's function,

$$\begin{aligned}&\frac{\hat{u}_\alpha^i(x)}{\int_M u_\alpha^{2^\#-1}dv_g} \xrightarrow{\alpha\to\infty} \frac{C\lambda_i}{|x|}+\beta_i(x) =: H_i \qquad \text{in } C^4_{loc}(B_0(\delta_0)\setminus\mathcal{S}),\\ &\text{with}\quad \lim_{x\to 0}\beta_i(x) = \lambda_i\mu_{x_i}(x_i)+\sum_{i\neq j}\lambda_j G(x_i,x_j)\end{aligned} \tag{IV.94}$$

for $i = 1,\dots N$.

Lemma 1.9. Under the assumptions and definitions above,

$$\lim_{r\to 0}\sup_{|x|=r}\sum_{k=1}^3 |x|^k\left|\nabla^k\beta_i(x)\right| = 0 \tag{IV.95}$$

for all $i = 1,\dots N$.

Proof. Since $\hat{u}_\alpha^i$ is defined as $\hat{u}_\alpha^i = \upsilon^i u_\alpha$, it solves the equation (IV.48), and thus by its definition β_i must also be solution to an equation of the type

$$\Delta^2\beta_i + A^{kl}\partial_k\partial_l\beta_i + B^k\partial_k\beta_i + D\beta_i = f_i \qquad \text{on } B_0(\delta_0)\setminus\{0\}, \tag{IV.96}$$

with smooth coefficients. To prove (IV.95), we start by assuming by contradiction that we have a sequence $(x_m)_m$ of points in $B_0(\delta_0)\setminus\{0\}$ such that

$$\sum_{k=1}^3 |x_m|^k\left|\nabla^k\beta_i(x_m)\right| \xrightarrow{m\to\infty} \infty. \tag{IV.97}$$

Since β_i is smooth on $\overline{B_0(\delta_0)} \setminus \{0\}$ and continuous at 0, one must necessarily have $x_m \to 0$. This leads to the definition of a sequence of functions $\beta_{i,m}(x) := \beta_i(|x_m|\, x)$. Then since β_i is continuous, $\beta_{i,m}$ must be bounded, and is defined on an increasingly larger ball as $m \to \infty$. Moreover, $\beta_{i,m}$ is solution to (IV.96) and so there exists $\overline{\beta}_i \in C^4(\mathbb{R}^n \setminus \{0\})$ such that up to a subsequence $\beta_{i,m} \to \overline{\beta}_i$ in $C^3_{loc}(\mathbb{R}^n \setminus \{0\})$ as $m \to \infty$. Since $x_m \to 0$, the $\beta_{i,m}$ are more and more zoomed versions of the β_i, and at the limit $\overline{\beta}_i$ should solve $\Delta^2 \overline{\beta}_i = 0$ in $\mathbb{R}^n \setminus \{0\}$ and $\left|\overline{\beta}_i\right| \leq C$.

One then gets

$$\sum_{k=1}^{3} |x_m|^k \left|\nabla^k \beta_i(x_m)\right| = \sum_{k=1}^{3} \left|\nabla^k \beta_{i,m}\left(\frac{x_m}{|x_m|}\right)\right| \xrightarrow{m\to\infty} \sum_{k=1}^{3} \left|\nabla^k \overline{\beta}_i(x_\infty)\right|, \tag{IV.98}$$

where x_∞ is the limit of $\left(\frac{x_m}{|x_m|}\right)_m$ as $m \to \infty$, $|x_\infty| = 1$, and we reached a contradiction with (IV.97). Thus we have for all $x \in B_0(\delta_0) \setminus \{0\}$,

$$\sum_{k=1}^{3} |x|^k \left|\nabla^k \beta_i(x)\right| \leq C. \tag{IV.99}$$

To conclude the proof, let us proceed once again by a contradiction argument, and suppose we have a sequence of (x_m) points in $B_0(\delta_0)$ with $x_m \to 0$ and

$$\sum_{k=1}^{3} |x_m|^k \left|\nabla^k \beta_i(x_m)\right| \geq C > 0 \qquad \forall\ m. \tag{IV.100}$$

We define $\beta_{i,m}$ as before, and $\overline{\beta}_i \in C^4(\mathbb{R}^n \setminus \{0\})$ such that $\beta_{i,m} \to \overline{\beta}_i$ in $C^3_{loc}(\mathbb{R}^n \setminus \{0\})$ with the same properties. Since $\Delta^2 \overline{\beta}_i = 0$ in $\mathbb{R}^n \setminus \{0\}$ and $\left|\overline{\beta}_i\right| \leq C$, for all $\varphi \in C^\infty_0(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \overline{\beta}_i \Delta^2 \varphi dx = \int_{|x|<\varepsilon} \overline{\beta}_i \Delta^2 \varphi dx + \int_{|x|\geq\varepsilon} \Delta^2 \overline{\beta}_i\ \varphi dx \leq C \int_{|x|<\varepsilon} \Delta^2 \varphi dx \xrightarrow{\varepsilon\to 0} 0. \tag{IV.101}$$

This means that $\Delta^2 \overline{\beta}_i = 0$ on the whole $\mathbb{R}^n$ in the distributional sense, since $\overline{\beta}_i \in C^4(\mathbb{R}^n \setminus \{0\})$, we must have $\overline{\beta}_i = Cst$ for some constant. But then as in (IV.98)

$$\sum_{k=1}^{3} |x_m|^k \left|\nabla^k \beta_i(x_m)\right| \xrightarrow{m\to\infty} \sum_{k=1}^{3} \left|\nabla^k \overline{\beta}_i(x_\infty)\right| = 0, \tag{IV.102}$$

and we reached once again a contradiction. □

We can now prove Theorem 1.8.

Proof (of Theorem 1.8). Assume that we have sequences (b_α), (c_α) of real numbers converging to b and c, and a sequence $(u_\alpha)_\alpha$ of smooth positive non-trivial solutions to (IV.82) that are bounded in $H^2(M)$ but that blow up. As in the discussion above we define $x_1, \dots x_N$ the concentration centers at the limit, and we have

$$\frac{\hat{u}^i_\alpha(x)}{\int_M u_\alpha^{2^\#-1} dv_g} \to H_i = \frac{C\lambda_i}{|x|} + \beta_i(x) \qquad \text{in } C^4_{loc}(M \setminus \mathcal{S}) \tag{IV.103}$$

with all the previous definitions. Then by the previous Lemma,

$$\lim_{r\to 0}\sup_{|x|=r}\sum_{k=1}^{3}|x|^k\left|\nabla^k\overline{\beta}_i(x)\right|=0 \qquad \text{for } i=1,\dots N. \tag{IV.104}$$

Using this, and applying the Pohozaev identity (IV.50) to $\hat{u}^i_\alpha$, for $\delta<\delta_0$

$$\int_{B_0(\delta)}\left(x^k\partial_k\hat{u}^i_\alpha+\frac{1}{2}\hat{u}^i_\alpha\right)\Delta^2\hat{u}^i_\alpha dx=\left(\int_M u_\alpha^{2^\#-1}dv_g\right)^2\left(A_\delta(H_i)+o(1)\right) \tag{IV.105}$$

where

$$\begin{aligned}A_\delta(H_i):=&-\frac{1}{2}\int_{\partial B_0(\delta)}\left(H_i\frac{\partial\Delta H_i}{\partial\nu}-\frac{\partial H_i}{\partial\nu}\Delta H_i\right)d\sigma+\frac{1}{2}\int_{\partial B_0(\delta)}(x,\nu)\,(\Delta H_i)^2d\sigma\\&-\int_{\partial B_0(\delta)}(x,\nabla H_i)\,\frac{\partial\Delta H_i}{\partial\nu}d\sigma+\int_{\partial B_0(\delta)}\frac{\partial\,(x,\nabla H_i)}{\partial\nu}\Delta H_i d\sigma,\end{aligned} \tag{IV.106}$$

and $\int_M u_\alpha^{2^\#-1}dv_g=O(\mu_\alpha)$ as in (IV.86).

But now with the main estimate Theorem III.2.1, since $u_\infty\equiv 0$,

$$\begin{aligned}\int_{B_0(\delta)}\hat{u}^i_\alpha\left|\nabla^k\hat{u}^i_\alpha\right|dx&\le\left(\int_{B_0(\delta)}(\hat{u}^i_\alpha)^2dx\right)^{\frac{1}{2}}\left(\int_{B_0(\delta)}\left|\nabla^k\hat{u}^i_\alpha\right|^2dx\right)^{\frac{1}{2}}\\&\le C(\delta)\mu_\alpha^{\frac{1}{2}}\mu_\alpha^{\frac{1}{2}}\end{aligned} \tag{IV.107}$$

and similarly

$$\int_{B_0(\delta)}\left|\nabla\hat{u}^i_\alpha\right|^2dx\le C(\delta)\mu_\alpha \tag{IV.108}$$

where $C(\delta)\xrightarrow{\delta\to 0}0$ and for $k=1,2,3$. Then as in the calculations in the proof of Proposition 1.5, integrating by part and using (IV.107,IV.108)

$$\left|\int_{B_0(\delta)}\left(x^k\partial_k\hat{u}^i_\alpha+\frac{1}{2}\hat{u}^i_\alpha\right)\Delta^2\hat{u}^i_\alpha dx\right|\le C(\delta)\mu_\alpha. \tag{IV.109}$$

With (IV.105), we get

$$\mu_\alpha(CA_\delta(H_i)+o(1))\le C(\delta)\mu_\alpha \tag{IV.110}$$

so that $A_\delta\to 0$ as $\delta\to 0$. But $H_i(x):=\frac{C\lambda_i}{|x|}+\beta_i(x)$ and thus we can compute $A_\delta(H_i)\xrightarrow{\delta\to 0}C\lambda_i\beta_i(0)$, with $\lambda_i>0$ for all i. Therefore, we must have

$$\beta_i(0)=\lambda_i\mu_{x_i}(x_i)+\sum_{i\neq j}\lambda_jG(x_i,x_j)=0, \tag{IV.111}$$

but if we start with the hypothesis that $\mu_x(x)>0$ for all $x\in M$, then $G>0$ and we have reached a contradiction. □

The case when $n = 6$. When the dimension of M is greater than 5, the stability of the Equation is intimately linked with the eigenvalues of the operator A_g. In the following we will consider $\lambda_i(A_g)_x$ the i-th eigenvalue of $A_g(x)$, and we will write $\lambda_1 := \inf_{x\in M} \inf_i \lambda_i(A_g)_x$ and $\lambda_2 := \sup_{x\in M} \sup_i \lambda_i(A_g)_x$. With these notations, all the eigenvalues of A_g will always be found in the interval $[\lambda_1, \lambda_2] =: \mathrm{Sp}_{A_g}$ called the *wild spectrum* of A_g.

Suppose we start with a sequence of $H^2(M)$-bounded solutions to (IV.82) such that the $(u_\alpha)_\alpha$ blow up. If $u_\infty \equiv 0$, then $u_\alpha \rightharpoonup 0$ in $H^2(M)$, and more precisely by the computations we did in the proof of Proposition II.4.2 we know that

$$\int_\Omega u_\alpha^2 dv_g \to 0, \qquad \int_\Omega |\nabla u_\alpha|^2 \, dv_g \to 0, \qquad \int_\Omega \Delta_g u_\alpha \Delta_g \varphi dv_g \to 0 \quad \forall\ \varphi \in C^\infty(M) \tag{IV.112}$$

and for all domain $\Omega \subseteq M$. However, since u_α blows up we also know it must concentrate in a shrinking ball around some centers, and in particular we expect all the L^2-norm of its gradient to be contained in a neighborhood of those bubble centers that we write $\mathcal{S} = \{\lim_{\alpha\to\infty} x_{i,\alpha} = x_i\}$. We will refer to Hebey, Robert, and Wen (2006) for the proof of the following.

Proposition 1.10. Let (M, g) be a smooth compact Riemannian manifold of dimension $n = 6, 7$, and $(u_\alpha)_\alpha$ a blowing-up sequence of non-trivial positive solutions to (IV.82). Assume $u_\infty \equiv 0$, then for all $\delta > 0$ the following holds, up to a subsequence

$$\frac{\int_{\cup_i B_{x_i}(\delta)} |\nabla u_\alpha|^2 \, dv_g}{\int_M |\nabla u_\alpha|^2 \, dv_g} \to 1. \tag{IV.113}$$

This property is called the ∇L^2-*concentration*. Moreover,

$$\int_M u_\alpha^2 dv_g = o(1) \int_M |\nabla u_\alpha|^2 \, dv_g, \tag{IV.114}$$

this is called the $L^2 - \nabla L^2$ *balance*.

This last property is the global equivalent of the Lemma 1.4 when $u_\infty \equiv 0$.

Corollary 1.10.i In the setting of the previous Proposition, for all $\delta > 0$, we have

$$\int_{M\setminus\cup_i B_{x_i}(\delta)} \left|\nabla^2 u_\alpha\right|^2 dv_g = o(1) \int_M |\nabla u_\alpha|^2 \, dv_g. \tag{IV.115}$$

Proof. We fix $\delta > 0$, then we take a smooth cut-off function around $M\setminus\cup_i B_{x_i}(\delta)$: Let $\eta \in C^\infty(M)$ be such that $\eta \equiv 1$ on $M \setminus \cup_i B_{x_i}(\delta)$ and $\eta \equiv 0$ on the sets $B_{x_i}(\delta/2)$, and we will write $\mathcal{B}(\delta) := \cup_i B_{x_i}(\delta)$. Since u_α is a weak solution to (IV.82), we can test it against $\eta^2 u_\alpha \in H^2(M)$ and get

$$\int_M \Delta_g u_\alpha \Delta_g(\eta^2 u_\alpha) dv_g + b_\alpha \int_M \left(\nabla u_\alpha, \nabla(\eta^2 u_\alpha)\right)_g dv_g + c_\alpha \int_M \eta^2 u_\alpha^2 dv_g = \int_M \eta^2 u_\alpha^{2^\#} dv_g. \tag{IV.116}$$

But now

$$\int_M \Delta_g u_\alpha \Delta_g(\eta^2 u_\alpha) dv_g = \int_M \left(\Delta_g(\eta u_\alpha)\right)^2 dv_g + O(\|u_\alpha\|^2_{H^1(\mathcal{B}(\delta)\setminus\mathcal{B}(\delta/2))}), \tag{IV.117}$$

where we recall $\|u_\alpha\|^2_{H^1(\Omega)} = \int_\Omega \left(|\nabla u_\alpha|^2 + u_\alpha^2\right) dv_g$. Similarly

$$\left(\nabla u_\alpha, \nabla(\eta^2 u_\alpha)\right)_M = \int_M |\nabla(\eta u_\alpha)|^2 \, dv_g + O(\|u_\alpha\|_{L^2(\mathcal{B}(\delta)\setminus\mathcal{B}(\delta/2))}) \tag{IV.118}$$

and we have the following identity that always holds

$$\int_M (\Delta_g(\eta u_\alpha))^2 \, dv_g = \int_M \left|\nabla^2(\eta u_\alpha)\right|^2 dv_g + O\left(\int_M |\nabla(\eta u_\alpha)|^2 \, dv_g\right). \tag{IV.119}$$

Independently, since $u_\alpha \to 0$ in $C^4_{loc}(M \setminus \mathcal{S})$, we have

$$\int_M \eta^2 u_\alpha^{2^\#} dv_g \leq \int_{M\setminus\mathcal{B}(\delta/2)} u_\alpha^{2^\#} dv_g = o\left(\int_{M\setminus\mathcal{B}(\delta/2)} u_\alpha^2 dv_g\right) = o\left(\int_M \eta^2 u_\alpha^2 dv_g\right), \tag{IV.120}$$

and so (IV.116) becomes

$$\begin{aligned}\int_M (\Delta_g(\eta u_\alpha))^2 dv_g + b_\alpha \int_M |\nabla(\eta u_\alpha)|^2 \, dv_g + (c_\alpha + o(1)) \int_M \eta^2 u_\alpha^2 dv_g \\ = O(\|u_\alpha\|^2_{L^2(\mathcal{B}(\delta)\setminus\mathcal{B}(\delta/2))}) + O(\|\nabla u_\alpha\|^2_{L^2(\mathcal{B}(\delta/2))}).\end{aligned} \tag{IV.121}$$

This implies

$$\begin{aligned}\int_{M\setminus\mathcal{B}(\delta)} \left|\nabla^2 u_\alpha\right|^2 dv_g \leq \int_M \left|\nabla^2(\eta u_\alpha)\right|^2 dv_g \leq \int_M (\Delta_g(\eta u_\alpha))^2 dv_g + C \int_M |\nabla(\eta u_\alpha)|^2 \, dv_g \\ \leq C_1 \int_M |\nabla(\eta u_\alpha)|^2 \, dv_g + C_2 \int_M (\eta u_\alpha)^2 dv_g \\ + C_3 \int_{\mathcal{B}(\delta)\setminus\mathcal{B}(\delta/2)} u_\alpha^2 dv_g + C_4 \int_{\mathcal{B}(\delta)\setminus\mathcal{B}(\delta/2)} |u_\alpha|^2 \, dv_g \\ \leq C_1 \int_{M\setminus\mathcal{B}(\delta/2)} u_\alpha^2 dv_g + C_2 \int_{M\setminus\mathcal{B}(\delta/2)} |u_\alpha|^2 \, dv_g,\end{aligned} \tag{IV.122}$$

where C_1, C_2 are two constants independent of α. Then using the previous proposition, we reach the conclusion

$$\int_{M\setminus\mathcal{B}} \left|\nabla^2 u_\alpha\right|^2 dv_g = o(1) \int_M |\nabla u_\alpha|^2 \, dv_g. \tag{IV.123}$$

□

Remark : The previous results hold actually for all dimensions $n \geq 6$, but will not need these other cases. The reader is referred to Hebey, Robert, and Wen (2006) for more details.

Theorem 1.11. Let (M, g) be a compact locally conformally flat manifold of dimension $n = 6$. Let $b, c > 0$ such that $\frac{b^2}{4} - c > 0$. If $b \notin \mathrm{Sp}_{A_g}$, then for every sequence of real numbers $(b_\alpha), (c_\alpha)$ converging respectively to b and c, any sequence $(u_\alpha)_\alpha$ of smooth non-trivial positive solutions to (IV.82) that are bounded in $H^2(M)$ admits a positive limiting solution u_∞ such that $u_\alpha \to u_\infty$ in $C^4(M)$.

Proof. We work by contradiction, and suppose we have such a sequence of solutions $(u_\alpha)_\alpha$ that blows up, with $b \notin \mathrm{Sp}_{A_g}$. By the stability Theorem 1.1 we must have $u_\infty \equiv 0$. As in the proof of Theorem 1.8 and since (M,g) is locally conformally flat, we can fix an asymptotic blowing up center x_i and identify it with 0. Let us take δ_0 and a function υ^i such that $g = (\upsilon^i)^{\frac{4}{n-4}}\xi$ is conformal to the Euclidean metric in a ball $B_0(\delta_0)$, with $B_0(\delta_0) \cap \mathcal{S} = \{x_i\}$. We also define $\hat{u}^i_\alpha = \upsilon^i u_\alpha$, then by (IV.113), (IV.114) and (IV.115), for $k = 1, 2, 3$ and all $0 < \delta < \delta_0$ we have

$$\int_{B_0(\delta)\setminus B_0(\delta/2)} \left|\nabla^k u_\alpha\right|^2 dx = o(1) \int_M |\nabla u_\alpha|^2 \, dv_g. \tag{IV.124}$$

As in (IV.55), $\hat{u}^i_\alpha$ solves

$$\begin{aligned}\Delta^2 \hat{u}^i_\alpha + b_\alpha \upsilon^{\frac{4}{n-4}}(-\Delta)\hat{u}^i_\alpha + B_\alpha(\nabla \upsilon^i, \nabla \hat{u}_\alpha) + C_\alpha \hat{u}^i_\alpha \\ + \upsilon^{\frac{n+4}{n-4}} \operatorname{div}_g(A_g \nabla(\upsilon^{-1}))\hat{u}^i_\alpha + \upsilon^{\frac{n+4}{n-4}} \operatorname{div}_g(\upsilon^{-1} A_g \nabla \hat{u}^i_\alpha) = (\hat{u}^i_\alpha)^{2^\#-1}.\end{aligned} \tag{IV.125}$$

Now we follow the arguments we used before, and apply the Pohozaev identity on $\eta_{(i)}\hat{u}^i_\alpha$ defining $\eta_{(i)}$ a smooth cut-off function such that $\eta \equiv 1$ on $B_0(\delta/2)$ and $\eta \equiv 0$ on $B_0(2\delta/3)$ so that all the boundary terms vanish – the index (i) will remind us that $\eta_{(i)}$ is a cut-off function around the center x_i when needed. We have on one hand

$$\begin{aligned}\int_{B_0(\delta)} \left(x^k \partial_k(\eta \hat{u}^i_\alpha) + \frac{n-4}{2}\eta \hat{u}^i_\alpha\right) \Delta^2(\eta \hat{u}^i_\alpha) dx = 0 \\ = \int_{B_0(\delta)} \eta^2 \left(x^k \partial_k \hat{u}^i_\alpha + \frac{n-4}{2}\hat{u}^i_\alpha\right) \Delta^2 \hat{u}^i_\alpha dx + o\left(\int_M |\nabla u_\alpha|^2 \, dv_g\right)\end{aligned} \tag{IV.126}$$

since all terms appearing involving η and u_α reduce to $o\left(\int_M |\nabla u_\alpha|^2 \, dv_g\right)$ by (IV.114) while all terms where a derivative of η appears are in fact integrals over $B_0(2\delta/3) \setminus B_0(\delta/2)$ and also go to zero faster than $\int_M |\nabla u_\alpha|^2 \, dv_g$ by (IV.124). Then we have

$$\begin{aligned}\int_{\mathbb{R}^n} \eta^2 \left(x^k \partial_k \hat{u}^i_\alpha + \frac{n-4}{2}\hat{u}^i_\alpha\right) \Delta^2 \hat{u}^i_\alpha dx = \int_{B_0(\delta)} \eta^2 \left(x^k \partial_k \hat{u}^i_\alpha + \frac{n-4}{2}\hat{u}^i_\alpha\right) \Delta^2 \hat{u}^i_\alpha dx \\ = o\left(\int_M |\nabla u_\alpha|^2 \, dv_g\right).\end{aligned} \tag{IV.127}$$

On the other hand $\hat{u}^i_\alpha$ solves (IV.125) so that as in the proof of Proposition 1.5

$$\begin{aligned}\int_{\mathbb{R}^n} \eta^2 \hat{u}^i_\alpha \Delta^2 \hat{u}^i_\alpha dx = \int_{\mathbb{R}^n} \eta^2 (\hat{u}^i_\alpha)^{2^\#} - b_\alpha \int_{\mathbb{R}^n} \eta^2 \upsilon^{\frac{4}{n-4}} \left|\nabla \hat{u}^i_\alpha\right|^2_\xi dx + \int_{\mathbb{R}^n} \eta^2 \upsilon^{\frac{8}{n-4}} A_g(\nabla \hat{u}^i_\alpha, \nabla \hat{u}^i_\alpha) dx \\ + o\left(\int_M |\nabla u_\alpha|^2 \, dv_g\right),\end{aligned} \tag{IV.128}$$

as all the other terms appearing when integrating by parts go inside the bound provided by o. For instance

$$\begin{aligned}\int_{\mathbb{R}^n} \eta^2 \upsilon^{\frac{4}{n-4}}(-\Delta)\hat{u}^i_\alpha = \int_{\mathbb{R}^n} \eta^2 \left(\nabla \hat{u}^i_\alpha, \nabla^i\right)_\xi \upsilon^{\frac{4}{n-4}} dx + 2\int_{\mathbb{R}^n} \eta \hat{u}^i_\alpha \left(\nabla \eta, \nabla \hat{u}^i_\alpha\right)_\xi \upsilon^{\frac{4}{n-4}} dx \\ + \frac{4}{n-4}\int_{\mathbb{R}^n} \eta^2 \hat{u}^i_\alpha \left(\nabla \upsilon, \nabla \hat{u}^i_\alpha\right)_\xi \upsilon^{\frac{8-n}{n-4}} dx,\end{aligned} \tag{IV.129}$$

then since

$$\int_{\mathbb{R}^n} \eta \hat{u}_\alpha^i \left(\nabla\eta, \nabla\hat{u}_\alpha^i\right)_\xi \upsilon^{\frac{4}{n-4}} dx = -\int_{\mathbb{R}^n} \eta \left(\nabla\eta, \nabla\hat{u}_\alpha^i\right)_\xi \hat{u}_\alpha^i \upsilon^{\frac{4}{n-4}} dx - \int_{\mathbb{R}^n} \hat{u}_\alpha^2 \left(\nabla\eta, \nabla\eta\right)_\xi \upsilon^{\frac{4}{n-4}} dx + \int_{\mathbb{R}^n} \eta (\hat{u}_\alpha^i)^2 (-\Delta)\eta \upsilon^{\frac{4}{n-4}} dx - \frac{4}{n-4} \int_{\mathbb{R}^n} \eta \hat{u}_\alpha^i \upsilon^{\frac{8-n}{n-4}} \left(\nabla\eta, \nabla\upsilon\right)_\xi dx \quad \text{(IV.130)}$$

we have in the end

$$\int_{\mathbb{R}^n} \eta \hat{u}_\alpha^i \left(\nabla\eta, \nabla\hat{u}_\alpha^i\right)_\xi \upsilon^{\frac{4}{n-4}} dx = O\left(\int_M u_\alpha^2 dv_g\right) = o\left(\int_M |\nabla u_\alpha|^2 dv_g\right), \quad \text{(IV.131)}$$

and identically for the other term.

Similarly we can compute

$$\int_{\mathbb{R}^n} \eta^2 (\Delta^2 \hat{u}_\alpha^i) x^k \partial_k \hat{u}_\alpha^i dx = -\frac{n-4}{2} \int_{\mathbb{R}^n} \eta^2 (\hat{u}_\alpha^i)^{2^\#} dx + b_\alpha \frac{n-2}{2} \int_{\mathbb{R}^n} \eta^2 \upsilon^{\frac{4}{n-4}} \left|\nabla\hat{u}_\alpha^i\right|^2 dx - \frac{n-2}{2} \int_{\mathbb{R}^n} \eta^2 \upsilon^{\frac{8}{n-4}} A_g(\nabla\hat{u}_\alpha^i, \nabla\hat{u}_\alpha^i) dx + C(\delta) \int_M |\nabla u_\alpha|^2 dv_g + o\left(\int_M |\nabla u_\alpha|^2 dv_g\right) \quad \text{(IV.132)}$$

where $C(\delta) \to 0$ as we let $\delta \to 0$. Putting all of this together with (IV.127),

$$b_\alpha \int_{\mathbb{R}^n} \eta_{(i)}^2 (\upsilon^i)^{\frac{4}{n-4}} \left|\nabla\hat{u}_\alpha^i\right|_\xi^2 dx - \int_{\mathbb{R}^n} \eta_{(i)}^2 (\upsilon^i)^{\frac{8}{n-4}} A_g(\nabla\hat{u}_\alpha^i, \nabla\hat{u}_\alpha^i) dx \leq o\left(\int_M |\nabla u_\alpha|^2 dv_g\right) + C(\delta) \int_M |\nabla u_\alpha|^2 dv_g. \quad \text{(IV.133)}$$

As before, we notice that $\left|\nabla\hat{u}_\alpha^i\right|_\xi^2 = \upsilon^{\frac{4}{n-4}} \left|\nabla\hat{u}_\alpha^i\right|_g^2$ so that

$$\left|\int_{\mathbb{R}^n} \eta_{(i)}^2 (\upsilon^i)^{\frac{8}{n-4}} (b_\alpha g - A_g)(\nabla\hat{u}_\alpha^i, \nabla\hat{u}_\alpha^i) dx\right| \leq C(\varepsilon_\delta + o(1)) \int_M |\nabla u_\alpha|^2 dv_g \quad \text{(IV.134)}$$

where $\varepsilon_\delta \xrightarrow{\delta\to 0} 0$, and this holds for all i.

Now when $b \notin \mathrm{Sp}_{A_g}$, $b_\alpha g - A_g$ is a bilinear operator either strictly positively or negatively defined for α big enough, and thus coming back to the global integral and using (IV.113),

$$\begin{aligned}
\int_M |\nabla u_\alpha|^2 dv_g &= \int_{\mathcal{B}(\delta/2)} |\nabla u_\alpha|^2 dv_g + o\left(\int_M |\nabla u_\alpha|^2 dv_g\right) \\
&= \sum_{i=1}^N \int_{B_0(\delta/2)} \left|\nabla\hat{u}_\alpha^i\right|^2 dx + o\left(\int_M |\nabla u_\alpha|^2 dv_g\right) \\
&\leq \sum_{i=1}^N \int_{\mathbb{R}^n} \eta_{(i)}^2 \left|\nabla\hat{u}_\alpha^i\right|^2 dx + o\left(\int_M |\nabla u_\alpha|^2 dv_g\right) \quad \text{(IV.135)} \\
&= \sum_{i=1}^N O\left(\int_{\mathbb{R}^n} \eta_{(i)}^2 (b_\alpha g - A_g)(\nabla\hat{u}_\alpha^i, \nabla\hat{u}_\alpha^i)(\upsilon^i)^{\frac{8}{n-4}} dx\right) + o\left(\int_M |\nabla u_\alpha|^2 dv_g\right) \\
&= C(\varepsilon_\delta + o(1)) \int_M |\nabla u_\alpha|^2 dv_g \qquad \forall\ 0 < \delta < \delta_0.
\end{aligned}$$

Since this holds true for all small enough $\delta > 0$, and that $\varepsilon \to 0$ as $\delta \to 0$, we reach a contradiction over the rate of decay between the left and the right hand terms. We conclude that $(u_\alpha)_\alpha$ cannot blow up in dimension $n = 6$ when $b \notin \mathrm{Sp}_{A_g}$. □

2 Sharp conditions for stability

In the previous section, we presented a stability condition in dimension $n = 6$, actually one can see that the proof of the Theorem 1.11 did not use the assumption on the dimension. Indeed, the condition $b \notin \mathrm{Sp}_{A_g}$ ensures stability for all $n \geq 6$ provided that $u_\alpha \rightharpoonup 0$. In this section, we will assume that the manifold we are dealing with is of dimension $n \geq 7$, this will allow to get new estimates on the key term $A_g(\nabla u_\alpha, \nabla u_\alpha)$ appearing in our equations. Then comparing the orders of convergence of the different quantities, we obtain sharper conditions for stability in higher dimensions.

Trace estimates. In dimensions $n \geq 7$, we will obtain more precise estimates on the bounds

$$\int_{B_{x_{i,\alpha}}(\delta\sqrt{\mu_\alpha})} A(\nabla u_\alpha, \nabla u_\alpha) dv_g = O\left(\int_M (\nabla u_\alpha, \nabla u_\alpha)\, dv_g\right) = O(\mu_\alpha^2). \tag{IV.136}$$

Since by Proposition III.1.2 we have that $R^{\mu_{i,\alpha}}_{x_{i,\alpha}} u_\alpha \to \mathrm{b}$ and that b has a radial symmetry, we expect to get $A_g(\nabla B_\alpha, \nabla B_\alpha) \propto \mu_\alpha^2 (A_g)^{ij} g_{ij} \, |\nabla \mathrm{b}|^2$, in other terms, we expect that the integral around the blow-up points behaves as the trace of A_g at first order.

We recall the expression of the operator,

$$A_g = \frac{(n-2)^2 + 4}{2(n-1)(n-2)} Sc_g g - \frac{4}{n-2} Ricc_g \tag{IV.137}$$

so that

$$\mathrm{Tr}_g\, A_g := \frac{(n-2)^2 + 4}{2(n-1)(n-2)} n Sc_g - \frac{4}{n-2} Sc_g = \frac{n^2 - 2n - 4}{2(n-1)} Sc_g. \tag{IV.138}$$

Assume $n \geq 7$, and take a sequence of $H^2(M)$-bounded positive non-trivial solutions $(u_\alpha)_\alpha$ that blow up. As before, we can define $\mu_\alpha := \max_{i=1,\dots K} \mu_{i,\alpha}$ and we can assume without loss of generality $\mu_\alpha = \mu_{1,\alpha}$, we also take $x_\alpha = x_{1,\alpha}$ the center associated to the lowest bubble. We will also need the subset of indices $I' := \{i \in \{1, \dots K\} \quad \text{s.t. } \mu_\alpha = O(\mu_{i,\alpha})\}$ and, for a fixed i, $\tilde{I}_i := \{j \in \{1, \dots K\} \quad \text{s.t. } d_g(x_{i,\alpha}, x_{j,\alpha}) = O(\mu_{i,\alpha})\}$, as well as the annuli

$$\mathcal{A}_{i,\alpha}(R) := B_{x_{i,\alpha}}(R\mu_{i,\alpha}) \setminus \cup_{j \in \tilde{I}_i} B_{x_{j,\alpha}}(\mu_{i,\alpha}/R). \tag{IV.139}$$

Lemma 2.1. Let $(\Omega_\alpha)_\alpha$ be a sequence of subsets of M, then

$$\int_{\Omega_\alpha \setminus \cup_{i \in I'} \mathcal{A}_{i,\alpha}(R)} |\nabla u_\alpha|^2 \, dv_g \leq 2 \int_{\Omega_\alpha} |\nabla u_\infty|^2 \, dv_g + o\left(\mathrm{Vol}_g(\Omega_\alpha)^{\frac{2}{n}}\right) + \varepsilon(R)\mu_\alpha^2 \tag{IV.140}$$

for all a, where $\varepsilon(R) \xrightarrow{R \to \infty} 0$.

In other words, when we remove only the neighborhoods of the bubbles blowing up at the same speed as B_{x_α,μ_α}, the only thing left in the L^2-norm of the gradient is essentially its weak limit.

Proof. We recall the initial decomposition in $H^2(M)$,

$$u_\alpha = u_\infty + \sum_{i=1}^{K} B_\alpha^i + R_\alpha, \qquad \|R_\alpha\|_{H^2} \to 0. \tag{IV.141}$$

Then we can write using Hölder's inequality

$$\begin{aligned}\int_{\Omega_\alpha \setminus \cup_{i\in I'} \mathcal{A}_{i,\alpha}(R)} |\nabla u_\alpha|^2 \, dv_g &\leq \int_{\Omega_\alpha \setminus \cup_{i\in I'} \mathcal{A}_{i,\alpha}(R)} |\nabla u_\infty|^2 \, dv_g + \sum_{j=1}^{K} \int_{\Omega_\alpha \setminus \cup_{i\in I'} \mathcal{A}_{i,\alpha}(R)} \left|\nabla B_\alpha^j\right|^2 dv_g \\ &\quad + \left(\int_{\Omega_\alpha} |\nabla R_\alpha|^{2^*} \, dv_g\right)^{\frac{n-2}{n}} (\mathrm{Vol}_g(\Omega_\alpha))^{\frac{2}{n}} \\ &\leq \int_{\Omega_\alpha} |\nabla u_\infty|^2 \, dv_g + \sum_{j=1}^{N} \int_{\Omega_\alpha \setminus \cup_{i\in I'} \mathcal{A}_{i,\alpha}(R)} \left|\nabla B_\alpha^j\right|^2 dv_g + o(1)\mathrm{Vol}_g(\Omega_\alpha)^{\frac{2}{n}} \end{aligned} \tag{IV.142}$$

by the continuous embedding $H^2(M) \hookrightarrow H^{1,2^*}(M)$ with $2^* = \frac{2n}{n-2}$,

$$\|\nabla R_\alpha\|_{L^{2^*}} \leq C \|R_\alpha\|_{H^{1,2^*}} \leq C \|R_\alpha\|_{H^2} \to 0 \tag{IV.143}$$

But for any fixed j,

$$\int_{\Omega_\alpha \setminus \cup_{i\in I'} \mathcal{A}_{i,\alpha}(R)} \left|\nabla B_\alpha^j\right|^2 dv_g \leq C\mu_{j,\alpha}^2 \int_{\mathbb{R}^n \setminus \frac{1}{\mu_{j,\alpha}} \exp_{x_{j,\alpha}}^{-1}(\mathcal{A}_\alpha(R))} \frac{|x|^2}{(1+|x|^2)^{n-2}} dx + o(\mu_\alpha^{n-4}) \tag{IV.144}$$

setting $\mathcal{A}_\alpha(R) := \cup_{i\in I'} \mathcal{A}_{i,\alpha}(R)$, and now $\mu_\alpha^{n-4} = o(\mu_\alpha^2)$ since $n \geq 7$.

Now we can treat separately in the sum of the $\int \left|\nabla B_\alpha^j\right|^2$ the $j \in I'$, such that $\mu_\alpha = O(\mu_{j,\alpha})$, and conversely the ones such that $j \notin I'$. In the latter case, we have by definition of μ_α and the structure property (II.153) $\mu_{j,\alpha} = o(\mu_\alpha^2)$ so that

$$\int_{\Omega_\alpha \setminus \mathcal{A}_\alpha(R)} \left|\nabla B_\alpha^j\right|^2 dv_g \leq C\mu_{j,\alpha}^2 \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^{n-2}} dx + o(\mu_\alpha^2) = o(\mu_\alpha^2). \tag{IV.145}$$

On the other hand when $j \in I'$, we have

$$\begin{aligned}\frac{1}{\mu_{j,\alpha}} \exp_{x_{j,\alpha}}^{-1}(\mathcal{A}_{i,\alpha}(R)) &= \frac{1}{\mu_{j,\alpha}} \exp_{x_{j,\alpha}}^{-1}(B_{x_{j,\alpha}}(R\mu_{j,\alpha}) \setminus \cup_{l\in\tilde{I}_j} B_{x_{l,\alpha}}(\frac{1}{R}\mu_{j,\alpha})) \\ &= B_0(R) \setminus \cup_{l\in\tilde{I}_j} B_{y_{l,\alpha}}(\frac{1}{R}) \end{aligned} \tag{IV.146}$$

where $y_{l,\alpha} := \frac{1}{\mu_{j,a}} \exp_{x_{j,\alpha}}^{-1}(x_{l,\alpha})$ and we also set $y_l := \lim_{\alpha\to\infty} y_{l,\alpha}$. Then integration over $\Omega_\alpha \setminus \frac{1}{\mu_{j,\alpha}} \exp_{x_{j,\alpha}}^{-1}(\mathcal{A}_\alpha(R))$ comes essentially to integrating removing a ball of radius R around 0, and

$$\begin{aligned}\int_{\Omega_\alpha \setminus \frac{1}{\mu_{j,\alpha}} \exp_{x_{j,\alpha}}^{-1}(\mathcal{A}_\alpha(R))} \left|\nabla B_\alpha^i\right|^2 dv_g &\leq C\mu_{j,\alpha}^2 \int_{\mathbb{R}^n \setminus \frac{1}{\mu_{j,\alpha}} \exp_{x_{j,\alpha}}^{-1}(\mathcal{A}_{j,\alpha}(R))} \frac{|x|^2}{(1+|x|^2)^{n-2}} dx \\ &\leq C\mu_{j,\alpha}^2 \left(\int_{\mathbb{R}^n \setminus (B_0(R) \setminus \cup_{l\in\tilde{I}_j} B_{y_k}(1/R))} \frac{|x|^2}{(1+|x|^2)^{n-2}} + o(1)\right) \\ &\leq C\mu_\alpha^2 \varepsilon(R) + o(\mu_\alpha^2) \end{aligned}$$

(IV.147)

with $\varepsilon(R) \to 0$ as $R \to \infty$.

Finally, we have the conclusion,

$$\int_{\Omega_\alpha \setminus \cup_{i\in I'} \mathcal{A}_{i\alpha}(R)} |\nabla u_\alpha|^2 \, dv_g \leq \int_{\Omega_\alpha} |\nabla u_\infty|^2 \, dv_g + o(\mathrm{Vol}_g(\Omega_\alpha)^{2/n}) + \varepsilon(R)\mu_\alpha^2 \tag{IV.148}$$

where $\varepsilon(R)$ also contains the $o(1)$ term. □

One then has a very similar estimate, when all the neighborhoods of all the bubbles are removed.

Lemma 2.2. With all the same previous hypothesis,

$$\int_{\Omega_\alpha \setminus \cup_{i=1}^K B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 \, dv_g \leq O(\mathrm{Vol}_g(\Omega_\alpha)) + \varepsilon(R)\mu_\alpha^2 \tag{IV.149}$$

where here too $\varepsilon(R) \xrightarrow{R\to\infty} 0$.

Proof. Using the main estimate Proposition III.2.1,

$$\begin{aligned}
\int_{\Omega_\alpha \setminus \cup_{i=1}^K B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 \, dv_g &\leq C \int_{\Omega_\alpha \setminus \cup_{i=1}^K B_{x_{i,\alpha}}(R\mu_\alpha)} (1 + \mu_\alpha^{n-4} r_\alpha^{6-2n}) dv_g \\
&\leq C\mathrm{Vol}_g(\Omega_\alpha) + C\mu_\alpha^{n-4} \sum_{i=1}^K \int_{\Omega_\alpha \setminus B_{x_{i,\alpha}}(R\mu_\alpha)} d_g(x_{i,\alpha}, x)^{6-2n} dv_g \\
&\leq C\mathrm{Vol}_g(\Omega_\alpha) + C\mu_\alpha^{n-4} \left(C_1 + C_2 \mu_\alpha^{6-n} \int_{\mathbb{R}^n \setminus B_0(R)} |x|^{6-2n} \, dx \right),
\end{aligned} \tag{IV.150}$$

and since $n \geq 7$, this last integral converges. We have Finally

$$\int_{\Omega_\alpha \setminus \cup_{i=1}^K B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 \, dv_g \leq O(\mathrm{Vol}_g(\Omega_\alpha)) + o(1) + C\mu_\alpha^2 \varepsilon(R) \tag{IV.151}$$

with $\varepsilon(R) \xrightarrow{R\to\infty} 0$. □

These two lemmas allow us to show new trace estimates, that hold for any smooth bilinear map. We define x_∞ as the limit of x_α the center of the main bubble, and the indices subsets

$$\begin{aligned}
I_1 &:= \{i \in \{1, \dots K\} \quad \text{s.t. } d(x_{i,\alpha}, x_\alpha) = o(1)\}, \\
I_2 &:= \{i \in \{1, \dots K\} \quad \text{s.t. } d(x_{i,\alpha}, x_\alpha) = o(\sqrt{\mu_\alpha})\}
\end{aligned} \tag{IV.152}$$

respectively the set of indices forming a cluster around x_α, and the set of indices that always remain in the interaction radius of B_α.

Proposition 2.3. Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 7$, and $b, c > 0$ such that $\frac{b^2}{4} - c > 0$. Let also $(b_\alpha), (c_\alpha)$ be sequences of real numbers that converge to b and c respectively, and $(u_\alpha)_\alpha$ a bounded sequence in $H^2(M)$ of positive non-trivial solutions to (IV.82) that blow up. If A is a smooth bilinear map vectors and $\delta > 0$ is such that $d_g(x_{i,\alpha}, x_\alpha) \geq 2\delta\sqrt{\mu_\alpha}$ for all α and all $i \notin I_2$, then there exists a constant $\Gamma > 0$ such that up to a subsequence

$$\lim_{\alpha\to\infty} \frac{1}{\mu_\alpha^2} \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} A(\nabla u_\alpha, \nabla u_\alpha) dv_g = \Gamma \frac{1}{n} \operatorname{Tr}_g(A)(x_\infty). \tag{IV.153}$$

Moreover, if $u_\infty \equiv 0$ and $\delta > 0$ is such that $d_g(x_{i,\alpha}, x_\alpha) \geq 2\delta$ for all α and all $i \notin I_1$, then there exists $\Gamma > 0$ such that

$$\lim_{\alpha\to\infty} \frac{1}{\mu_\alpha^2} \int_{B_{x_\alpha}(\delta)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g = \Gamma \frac{1}{n} \operatorname{Tr}_g(A)(x_\infty). \tag{IV.154}$$

Proof. As we did before, we write $\mathcal{A}_\alpha(R) := \cup_{i\in I'} \mathcal{A}_{i,\alpha}(R)$ and $\mathcal{B}_\alpha(R) := \cup_{i=1}^K B_{x_{i,\alpha}}(R\mu_\alpha)$, by the structure equation (II.153) we know that $\mathcal{A}_{i,\alpha} \cap \mathcal{A}_{j,\alpha} = \emptyset$ for all $i \neq j$ in I'. Writing $J := I' \cap I_2$, the set of all indices of the bubbles that remain in the interaction radius of B_α and blow up at the same rate, then

$$\begin{aligned}\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} A(\nabla u_\alpha, \nabla u_\alpha) dv_g &= \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\cap\mathcal{A}_\alpha(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g \\ &\quad + \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\setminus\mathcal{A}_\alpha(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g \\ &= \sum_{i\in J} \int_{\mathcal{A}_{i,\alpha}(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g + \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\setminus\mathcal{A}_\alpha(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g. \end{aligned} \tag{IV.155}$$

But

$$\begin{aligned}\left|\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\setminus\mathcal{A}_\alpha(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g\right| &\leq \left|\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\setminus\mathcal{B}_\alpha(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g\right| \\ &\quad + \left|\int_{\mathcal{B}_\alpha(R)\setminus\mathcal{A}_\alpha(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g\right| \end{aligned} \tag{IV.156}$$

and by Lemma 2.2,

$$\begin{aligned}\left|\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})\setminus\mathcal{B}_\alpha(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g\right| &\leq C(\delta\sqrt{\mu_\alpha})^n + \varepsilon(R)\mu_\alpha^2 \\ &= o(\mu_\alpha^2) + \varepsilon(R)\mu_\alpha^2 \end{aligned} \tag{IV.157}$$

where $\varepsilon(R) \to 0$ as $R \to \infty$.

Using now Lemma 2.1,

$$\begin{aligned}\left|\int_{\mathcal{B}_\alpha(R)\setminus\mathcal{A}_\alpha(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g\right| &\leq \int_{\cup_{i=1}^K B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2 \, dv_g + o((R\mu_\alpha)^{n\,\frac{2}{n}}) + \varepsilon(R)\mu_\alpha^2 \\ &\leq C\mu_\alpha^{n-2} + o(\mu_\alpha^2) + \varepsilon(R)\mu_\alpha^2, \end{aligned} \tag{IV.158}$$

and $\mu_\alpha^{n-2} = o(\mu_\alpha^2)$. This gives

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} A(\nabla u_\alpha, \nabla u_\alpha) dv_g = \sum_{i\in J} \int_{\mathcal{A}_{i,\alpha}(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g + o(\mu_\alpha^2) + \varepsilon(R)\mu_\alpha^2 \tag{IV.159}$$

with $\varepsilon(R) \xrightarrow{R\to\infty} 0$.

We now use the Proposition III.1.2, giving the convergence of u_α to an Euclidean bubble when looked at a specific scale, to estimate the integrals that appear in the sum. To do that we set $\tilde{u}_\alpha := R^{\mu_{i,\alpha}}_{x_{i,\alpha}} u_\alpha$ for a fixed $i \in J$, this is a function defined on $B_0(i_g(M)/(2\mu_{i,\alpha}))$ for all α. We need also to define $g_\alpha(x) := (\exp^*_{x_{i,\alpha}} g)(\mu_{i,\alpha} x)$ and for all the bubbles that will remain visible in the zoom, i.e. all $j \in \tilde{I}_i = \{j = 1, \dots K \quad \text{s.t. } d_g(x_{i,\alpha}, x_{j,\alpha}) = O(\mu_{i,\alpha})\}$ we write $y_{j,\alpha} := \frac{1}{\mu_{i,\alpha}} \exp^{-1}_{x_{i,\alpha}}(x_{j,\alpha})$. Finally, writing $A_\alpha(x) := (\exp^*_{x_{i,\alpha}} A)(\mu_{i,\alpha} x)$ we have

$$\int_{\mathcal{A}_{i,\alpha}(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g = \mu_{i,\alpha}^2 \int_{B_0(R)\setminus \cup_{j\in\tilde{I}_i} B_{y_{j,\alpha}}(1/R)} A_\alpha(\nabla \tilde{u}_\alpha, \nabla \tilde{u}_\alpha) dv_{g_\alpha} \tag{IV.160}$$

where the balls are taken in the metric g_α, but since $g_\alpha \to \xi$ in $C^4_{loc}(\mathbb{R}^n)$, and by Proposition III.1.2,

$$\int_{B_0(R)\setminus \cup_{j\in\tilde{I}_i} B_{y_{j,\alpha}}(1/R)} A_\alpha(\nabla \tilde{u}_\alpha, \nabla \tilde{u}_\alpha) dv_{g_\alpha} \xrightarrow{\alpha\to\infty} \int_{B_0(R)\setminus \cup_{j\in\tilde{I}_i} B_{y_j}(1/R)} A_\infty(\nabla \mathrm{b}, \nabla \mathrm{b}) dx \tag{IV.161}$$

with $B_{y_j}(1/R)$ is an Euclidean ball of center $y_j = \lim_{\alpha\to\infty} y_{j,\alpha}$ and radius $1/R$, and $A_\infty = (\exp^*_{x_{i,\infty}} A)(0)$. Since $i \in J$, we have $x_{i,\infty} = x_\infty$, and since A_∞ is a bounded operator, using the explicit expression of b the integral converges and finally

$$\begin{aligned}
\int_{\mathcal{A}_{i,\alpha}(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g &= \mu_{i,\alpha}^2 \left(\int_{B_0(R)\setminus \cup_{j\in\tilde{I}_i} B_{y_j}(1/R)} A_\infty(\nabla \mathrm{b}, \nabla \mathrm{b}) dx + o(1) \right) \\
&= \mu_{i,\alpha}^2 \int_{\mathbb{R}^n} A_\infty(\nabla \mathrm{b}, \nabla \mathrm{b}) dx + o(\mu_\alpha^2) + \varepsilon(R)\mu_{i,\alpha}^2, \\
&= \mu_\alpha^2 \frac{1}{n} \operatorname{Tr}_g(A)(x_\infty) \left(\frac{\mu_{i,\alpha}}{\mu_\alpha}\right)^2 \int_{\mathbb{R}^n} |\nabla \mathrm{b}|^2 \, dx + o(\mu_\alpha^2) + \varepsilon(R)\mu_\alpha^2
\end{aligned} \tag{IV.162}$$

since $i \in J$ and that b has a radial symmetry, with as always $\varepsilon(R) \xrightarrow{R\to\infty} 0$.

By setting $\tilde{\mu}_i := \lim_{\alpha\to\infty} \frac{\mu_{i,\alpha}}{\mu_\alpha}$ which converges for all $i \in I' = \{i \mid \mu_\alpha = O(\mu_{i,\alpha})\} \subset J$ and then $\Gamma = \left(\sum_{i\in J} \tilde{\mu}_i^2 \int_{\mathbb{R}^n} |\nabla \mathrm{b}|^2 \, dx\right)$, we know $\Gamma > 0$ since $1 \in J$ and we proved the first affirmation.

Let us now prove the second one, when $u_\infty \equiv 0$, then using the main estimation Theorem III.2.1,

$$\begin{aligned}\int_{B_{x_\alpha}(\delta)\setminus\mathcal{B}_\alpha(R)} |\nabla u_\alpha|^2 \, dv_g &\leq C\mu_\alpha^{n-4} \int_{B_{x_\alpha}(\delta)\setminus\mathcal{B}_\alpha(R)} r_\alpha(x)^{6-2n} dv_g \\ &\leq C\mu_\alpha^{n-4} \sum_{i=1}^K \int_{B_{x_\alpha}(\delta)\setminus B_{x_{i,\alpha}}(R\mu_\alpha)} d_g(x_{i,\alpha}, x)^{6-2n} dv_g \\ &\leq C\mu_\alpha^{n-4} \sum_{i=1}^K \int_{B_{\frac{1}{\mu_\alpha}\exp_{x_{i,\alpha}}^{-1}(x_\alpha)}(\delta/\mu_\alpha)\setminus B_0(R)} \mu_\alpha^{6-2n} |y|^{6-2n} \mu_\alpha^n dy \qquad \text{(IV.163)} \\ &\leq C\mu_\alpha^{n-4} \sum_{i=1}^K \left(\mu_\alpha^{6-n} \int_{\mathbb{R}^n\setminus B_0(R)} |y|^{6-2n} \, dy \right) \\ &= \mu_\alpha^2 \varepsilon(R)\end{aligned}$$

with $\varepsilon(R) \to 0$ as $R \to \infty$ since $n \geq 7$. But now

$$\int_{B_{x_\alpha}(\delta)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g = \sum_{i\in J'} \int_{\mathcal{A}_{i,\alpha}(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g + \int_{B_{x_\alpha}(\delta)\setminus\mathcal{A}_\alpha(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g \quad \text{(IV.164)}$$

where we defined $J' := I' \cap I_1$ the set of indices of the bubbles that cluster around B_α and blow up at the same rate. As in the first part of the proof, we have

$$\begin{aligned}\left| \int_{B_{x_\alpha}(\delta)\setminus\mathcal{A}_\alpha(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g \right| &\leq \left| \int_{B_{x_\alpha}(\delta)\setminus\mathcal{B}_\alpha(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g \right| \\ &\qquad + \left| \int_{\mathcal{B}_\alpha(R)\setminus\mathcal{A}_\alpha(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g \right| \qquad \text{(IV.165)} \\ &\leq C\delta^n + \varepsilon(R)\mu_\alpha^2 + o(\mu_\alpha^2).\end{aligned}$$

So if we take $\delta \leq d_g(x_{i,\alpha}, x_\alpha)$ for all $i \notin I_1$, we have

$$\int_{B_{x_\alpha}(\delta)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g = \sum_{i\in J'} \int_{\mathcal{A}_{i,\alpha}(R)} A(\nabla u_\alpha, \nabla u_\alpha) dv_g + o(\mu_\alpha^2) + \varepsilon(R)\mu_\alpha^2 \quad \text{(IV.166)}$$

and using the same arguments as in (IV.162) we reach the conclusion in this case too. □

Remark : Note that we have an explicit expression for the constant Γ, using the estimate with the metric operator, $\mathrm{Tr}_g\, g = n$ so that

$$\lim_{\alpha\to\infty} \frac{1}{\mu_\alpha^2} \int_{B_{x_\alpha}(\delta\sqrt{\mu_a})} (\nabla u_\alpha, \nabla u_\alpha)_g \, dv_g = \Gamma. \quad \text{(IV.167)}$$

This expression should be equivalent to the definition of Γ we got in the proof, and it can be verified following the same steps for the operator $A = g^{-1}$.

Stability in all dimensions. As we already saw at the end of the previous section, we have a stability condition in dimensions $n \geq 6$, that is $b \notin \mathrm{Sp}_{A_g}$, provided that $u_\infty \equiv 0$. This was proved already in Hebey, Robert, and Wen (2006). In the main article, Hebey and Robert (2011) show less restrictive conditions for higher dimensions, relying on the principal estimates' Theorem and these trace estimates. It is clear that since the trace of the operator is an invariant, $\frac{1}{n}\operatorname{Tr}_g A_g \in \mathrm{Sp}_{A_g}$ at each point. Indeed, if one takes the basis of eigenvectors of A_g, then $\operatorname{Tr}_g A_g = \sum_i \lambda_i(A_g)$ so that $\frac{1}{n}\operatorname{Tr}_g A_g$ is the mean of its eigenvalues and $\lambda_1 \leq \frac{1}{n}\operatorname{Tr}_g A_g(x) \leq \lambda_2$ for all $x \in M$. We have the following Theorem.

Theorem 2.4. Let (M, g) be a compact locally conformally flat manifold and $b, c > 0$ such that $\frac{b^2}{4} - c > 0$.

i. If $\dim M = 8$ and $b < \frac{1}{n}\min_{x\in M} \operatorname{Tr}_g A_g(x)$, or

ii. if $\dim M = 7$ or $\dim M \geq 9$ and $b \neq \frac{1}{n}\operatorname{Tr}_g A_g(x)$ for all $x \in M$,

then for every sequence of real numbers $(b_\alpha), (c_\alpha)$ converging respectively to b and c, and any sequence $(u_\alpha)_\alpha$ of smooth non-trivial positive solutions to (IV.82) that are bounded in $H^2(M)$, (u_α) converges to $u_\alpha \to u_\infty$ in $C^4(M)$.

Proof (when $\dim = 7$). In the case of dimension $n = 7$, we already know by Theorem 1.1 that $u_\infty \equiv 0$ if we assume the sequence u_α to blow up. We can then come back to (IV.134) in the proof of Theorem 1.11 around x_α, center associated to the lowest bubble, i.e. taking $\mu_\alpha = \max_{i=1,\dots K} \mu_{i,\alpha} = \mu_{1,\alpha}$ for all α without loss of generality and defining $x_\alpha := x_{1,\alpha}$. We can identify x_α to 0 and take υ function such that $g = \upsilon^{\frac{4}{n-4}}\xi$ in $B_{x_\alpha}(\delta)$ for a small enough δ. Defining $\hat{u}_\alpha = \upsilon u_\alpha$, we have

$$\left|\int_{\mathbb{R}^n} \eta^2 \upsilon^{\frac{8}{n-4}}(A_g - b_\alpha g)(\nabla\hat{u}_\alpha, \nabla\hat{u}_\alpha)dx\right| \leq (C(\delta) + o(1))\int_M |\nabla u_\alpha|^2\, dv_g \tag{IV.168}$$

where η is a cut-off function around 0 and $C(\delta) \to 0$ as $\delta \to 0$.

But using (IV.113),

$$\int_M |\nabla u_\alpha|^2\, dv_g = \sum_{i=1}^K \int_{B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2\, dv_g + o(1) \tag{IV.169}$$

and as in the proof of the Lemma 1.6, (IV.77), thanks to Hölder's inequality

$$\int_{B_{x_{i,\alpha}}(R\mu_\alpha)} |\nabla u_\alpha|^2\, dv_g \leq C\mu_\alpha^{n\left(1-\frac{2(n-2)}{2n}\right)} = C\mu_\alpha^2 \tag{IV.170}$$

by the embedding $H^1(M) \hookrightarrow L^{\frac{2n}{n-2}}(M)$. Using the global estimates, since $n = 7$, $|\nabla u_\alpha|^2$ is integrable on $B_{x_{i,\alpha}}(R\mu_\alpha)$, and the trace estimates give

$$\frac{1}{\mu_\alpha^2}\int_{B_{x_\alpha}(\delta)} A_g(\nabla u_\alpha, \nabla u_\alpha)dv_g = \frac{\Gamma}{n}\operatorname{Tr}_g A_g(x_\infty). \tag{IV.171}$$

Letting $\delta \to 0$ in (IV.168),

$$\int_{\mathbb{R}^n} \eta^2 \upsilon^{\frac{8}{n-4}} A_g(\nabla\hat{u}_\alpha, \nabla\hat{u}_\alpha)dx - b_\alpha \int_{\mathbb{R}^n} \eta^2 \upsilon^{\frac{8}{n-4}} |\nabla\hat{u}_\alpha|_g^2\, dx \leq o(\mu_\alpha^2), \tag{IV.172}$$

and thus when $\alpha \to \infty$, by the Remark (IV.167)

$$\mathrm{Tr}_g\, A_g(x_\infty) - n\, b = o(1). \tag{IV.173}$$

Assuming $b \neq \frac{1}{n}\,\mathrm{Tr}_g\, A_g(x_\infty)$, we reach a contradiction, proving our stability theorem. □

Proof (when dim ≥ 8). If we assume that (u_α) blows up, in this case $n \geq 8$, we don't know necessarily whether $u_\infty \equiv 0$ and we have to come back to a local description at distance $\sim \sqrt{\mu_\alpha}$. We take back $\mu_\alpha = \max_{i=1,\dots K} \mu_{i,\alpha}$ and x_α the associated center, and the proof of Lemma 1.6 where we proved that

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2\, dv_g \leq C\mu_\alpha^2 + o(\mu_\alpha^{\frac{n-4}{2}}) \qquad \forall \delta > 0. \tag{IV.174}$$

Since $n \geq 8$ one cannot say that $\mu_\alpha^2 = o(\mu_\alpha^{\frac{n-4}{2}})$, but by Proposition 1.5,

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} (A_g - b_\alpha g)(\nabla u_\alpha, \nabla u_\alpha) dv_g = o\left(\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2\, dv_g\right) - (K(u_\infty)+o(1))\mu_\alpha^{\frac{n-4}{2}} \tag{IV.175}$$

and by the trace estimates

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} A_g(\nabla u_\alpha, \nabla u_\alpha) dv_g = \frac{\Gamma}{n}\,\mathrm{Tr}_g\, A_g(x_\infty)\mu_\alpha^2 + o(\mu_\alpha^2). \tag{IV.176}$$

This gives, letting $\alpha \to \infty$,

$$(\frac{1}{n}\,\mathrm{Tr}_g\, A_g(x_\infty) - b + o(1))\mu_\alpha^2 = -\frac{1}{\Gamma}(K(u_\infty) + o(1))\mu_\alpha^{\frac{n-4}{2}}. \tag{IV.177}$$

Now if $n \geq 9$, $\mu_\alpha^{\frac{n-4}{2}} = o(\mu_\alpha^2)$ and we reach a contradiction if $b \neq \frac{1}{n}\,\mathrm{Tr}_g\, A_g(x_\infty)$, and if $n = 8$ then $\mu_\alpha^{\frac{n-4}{2}} = \mu_\alpha^2$ and we also get a contradiction when $b < \frac{1}{n}\,\mathrm{Tr}_g\, A_g$ because of the sign of the left and right-hand side in the equality as $\alpha \to \infty$. □

Remark : We take back the Example I.1.3.ii, where we expressed the Paneitz equation in an Einstein metric. If $Ricc_g = \lambda g$, then by definition $|Ricc_g|_g^2 = \lambda^2 n$ and

$$Pan_g u = \Delta_g^2 u + \lambda b_n \Delta_g u + c_n \lambda^2 u, \tag{IV.178}$$

with

$$b_n = \frac{n^2 - 2n - 4}{2(n-1)} \qquad c_n = \frac{n(n-4)(n^2-4)}{16(n-1)^2}. \tag{IV.179}$$

This last Theorem ensures that the equation $P_g u = u^{2^\# - 1}$ is stable whenever the term multiplying Δ_g is never equal to $\frac{1}{n}\,\mathrm{Tr}_g\, A_g(x)$ for all $x \in M$, with

$$\mathrm{Tr}_g\, A_g(x) = \frac{n^2 - 2n - 4}{2(n-1)} Sc_g(x). \tag{IV.180}$$

But in an Einstein metric, one has immediately $Sc_g = g^{ij}(Ricc_g)_{ij} = \lambda n$, and we recognise exactly the term appearing in the Paneitz operator. This means that actually, a critical bi-Laplacian equation of the shape

$$P_g u = \Delta_g^2 u + b\Delta_g u + cu = u^{2^\#-1} \tag{IV.181}$$

is stable unless it is too close to the conformal transformation equation of Q-curvature on an Einstein locally conformally flat Riemannian manifold. This was already noticed on the sphere which is indeed locally conformally flat. In that case we saw in Example II.1.4, bubbles appearing naturally as conformal terms in the transformation of the metric through the inverse stereographic projection.

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